An introduction to separable equivalence

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Beginning note

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algebra finitely generated unital algebra over k

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functor	means k-linear functor

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$$F\colon \operatorname{mod} A \longrightarrow \operatorname{mod} B \qquad \mathrm{Id}_{\operatorname{mod} A} \cong GF$$

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In particular $\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_B(FM, FN)$.

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Example (2)

Any algebra A is Morita equivalent to the $n \times n$ matrix ring $\mathbb{M}_n(A)$.

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Sketch proof.

Given the modules M and N, the functors

$$- \underset{A}{\otimes} M \colon \operatorname{mod} A \to \operatorname{mod} B \qquad - \underset{B}{\otimes} N \colon \operatorname{mod} B \to \operatorname{mod} A$$

provide an equivalence of the categories.

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- kA_5 is stably equivalent to kA_4 but not Morita equivalent

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Thus

- kA_5 is Morita equivalent to $\mathbb{M}_n(kA_5)$ but not isomorphic
- kA_5 is stably equivalent to kA_4 but not Morita equivalent
- kA_5 is separably equivalent to $k[C_2 \times C_2]$ but not stably equivalent

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Proposition

Separable equivalence preserves complexity.

Question

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$$\operatorname{cx} kG = \max \left\{ r \;\middle|\; \overbrace{C_p \times \cdots \times C_p}^{r} \leq G \right\}$$

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Slightly longer answer

For G a group

$$\mathrm{cx}\, kG = \max\left\{ r \,\middle|\, \overbrace{C_p \times \cdots \times C_p}^{r} \leq G \right\}$$

So $kC_p, kC_{p^2}, kC_{p^3}, \ldots$ all have complexity 1.

New question

Is kC_{p^n} separably equivalent to kC_{p^m} for any $m \neq n$?

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Short answer

Maybe.

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${}^{k[x]}\big/_{(x^2)}$ is not separably equivalent to ${}^{k[y]}\big/_{(y^n)}$ for $n \neq 2$.

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This is about as far from a proof as I'm going to get.

If

$$\Lambda_2 \overset{sep}{\sim} \Lambda_n$$

Slightly longer answer

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If

$$\Lambda_2 \overset{sep}{\sim} \Lambda_n$$

then

$$\mathrm{Fun}(\underline{\mathrm{mod}}\,\Lambda_2, \mathrm{vec}\,) \overset{sep}{\sim} \mathrm{Fun}(\underline{\mathrm{mod}}\,\Lambda_n, \mathrm{vec}\,)$$

Slightly longer answer

$k[x]/(x^2)$ is not separably equivalent to $k[y]/(y^n)$ for $n \neq 2$.

This is about as far from a proof as I'm going to get.

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$$\mathrm{Fun}(\underline{\mathrm{mod}}\,\Lambda_2, \mathrm{vec}\,) :$$

$\square$

Slightly longer answer

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$$\mathrm{Fun}(\underline{\mathrm{mod}}\,\Lambda_2, \mathrm{vec}\,) : \qquad \bullet$$

$\square$

Slightly longer answer

$k[x] / (x^2)$ is not separably equivalent to $k[y] / (y^n)$ for $n \neq 2$.

This is about as far from a proof as I'm going to get.

If

$$\Lambda_2 \overset{sep}{\sim} \Lambda_n$$

then

$$\mathrm{Fun}(\underline{\mathrm{mod}}\, \Lambda_2, \mathrm{vec}) \overset{sep}{\sim} \mathrm{Fun}(\underline{\mathrm{mod}}\, \Lambda_n, \mathrm{vec})$$

$$\mathrm{Fun}(\underline{\mathrm{mod}}\, \Lambda_2, \mathrm{vec}) : \qquad \bullet$$

$$\mathrm{Fun}(\underline{\mathrm{mod}}\, \Lambda_n, \mathrm{vec}) :$$

$\square$

Slightly longer answer

$k[x]/_{(x^2)}$ is not separably equivalent to $k[y]/_{(y^n)}$ for $n \neq 2$.

This is about as far from a proof as I'm going to get.

If

$$\Lambda_2 \overset{sep}{\sim} \Lambda_n$$

then

$$\mathrm{Fun}(\underline{\mathrm{mod}}\,\Lambda_2, \mathrm{vec}\,) \overset{sep}{\sim} \mathrm{Fun}(\underline{\mathrm{mod}}\,\Lambda_n, \mathrm{vec}\,)$$

$$\mathrm{Fun}(\underline{\mathrm{mod}}\,\Lambda_2, \mathrm{vec}\,) : \qquad \bullet$$

$$\mathrm{Fun}(\underline{\mathrm{mod}}\,\Lambda_n, \mathrm{vec}\,) : \quad \begin{tikzcd} 1 \arrow[r, bend left] & 2 \arrow[l, bend left] \arrow[r, bend left] & 3 \arrow[l, bend left] \arrow[r, dashed, bend left] & {} \arrow[l, dashed, bend left] & {} \arrow[r, bend left] & n-1 \arrow[l, bend left] \end{tikzcd}$$

with some relations

$\square$

Ta.