DECOMPOSITION ALGEBRAS AND AXIAL ALGEBRAS

TOM DE MEDTS, SIMON F. PEACOCK, SERGEY SHPECTOROV, AND MICHIEL VAN COUWENBERGHE

Abstract. We introduce decomposition algebras as a natural generalization of axial algebras, Majorana algebras and the Griess algebra. They remedy three limitations of axial algebras: (1) They separate fusion laws from specific values in a field, thereby allowing repetition of eigenvalues; (2) They allow for decompositions that do not arise from multiplication by idempotents; (3) They admit a natural notion of homomorphisms, making them into a nice category.

We exploit these facts to strengthen the connection between axial algebras and groups. In particular, we provide a definition of a universal Miyamoto group which makes this connection functorial under some mild assumptions.

We illustrate our theory by explaining how representation theory and association schemes can help to build a decomposition algebra for a given (permutation) group. This construction leads to a large number of examples.

We also take the opportunity to fix some terminology in this rapidly expanding subject.

1. Introduction

In 1982, Robert Griess proved the existence of the Monster group by constructing a 196 884-dimensional non-associative algebra over $\mathbb{R}$, called the *Griess algebra* [Gri82]. A peculiar feature of these algebras is the existence of many idempotents with the property that multiplication by each of these idempotents gives rise to a *decomposition* of the algebra obeying a very precise *fusion law*.

Igor Frenkel, James Lepowsky and Arne Meurmann observed that other algebras similar the Griess algebra can be retrieved as weight-2 components of certain *vertex operator algebras* (VOAs) [FLM88]. In an attempt to axiomatize such algebras, Alexander Ivanov introduced *Majorana algebras*, a large class of real non-associative algebras obeying the same fusion law as the Griess algebra.

Only recently, in 2015, the more general concept of *axial algebras* was introduced by Jonathan Hall, Sergey Shpectorov and Felix Rehren [HRS15a]. Axial algebras are defined over an arbitrary field and have as defining feature that they are generated by idempotents that again give rise to decompositions satisfying a fusion law, which is now allowed to take a much more general shape. The subject has received a lot of attention since then and developed connections as far afield as the regularity theory of some classes of elliptic type PDEs and algebraic solutions of eiconal and minimal surface equations [Tka19a, Tka19b]. See also the earlier book [NTV14].

In May 2018, a specialized workshop on axial algebras took place at the University of Bristol funded by the Heilbronn Institute for Mathematical Research. It became apparent at this workshop that there is a need for a more general framework to study axial algebras. New observations forced us to generalize the definition even further and to separate fusion laws from the field. At the same time, we noticed that the crucial aspect of an axial algebra is the existence of the corresponding decompositions, and not so much the fact that these arise from idempotents.

[Gri82] Griess, Jr., *The friendly giant*, Invent. Math. **69** (1982), no. 1, 1–102

[FLM88] Frenkel, Lepowsky, and Meurman, *Vertex operator algebras and the Monster*, Pure and Applied Mathematics, vol. 134, Academic Press, Inc., Boston, MA, 1988

[HRS15a] Hall, Rehren, and Shpectorov, *Primitive axial algebras of Jordan type*, J. Algebra **437** (2015), 79–115

[Tka19a] Tkachev, *Spectral properties of nonassociative algebras and breaking regularity for nonlinear elliptic type pdes*, Algebra I Analiz **31** (2019), no. 2, 51–74

[Tka19b] Tkachev, *V.M. Miklyukov: from dimension 8 to nonassociative algebras*, Math. Phys. Comp. Simul. **22** (2019), no. 2, To appear; available on arxiv:1904.08162

[NTV14] Nadirashvili, Tkachev, and Vlăduţ, *Nonlinear elliptic equations and nonassociative algebras*, Mathematical Surveys and Monographs, vol. 200, American Mathematical Society, Providence, RI, 2014

Date: November 7, 2019.

Key words and phrases. decomposition algebras, axial algebras, fusion laws, Griess algebra, Majorana algebras, representation theory, association schemes, Norton algebras.

The *decomposition algebras* that we introduce in this paper aim to provide a natural generalization of axial algebras that take all these facts into account. Our hope that this is a useful framework is further emphasized by the fact that these decomposition algebras form a nice category (in contrast to the setting of axial algebras, where the natural notion of homomorphisms gives rise to a less powerful category).

We begin our paper by introducing (general) *fusion laws* that no longer depend on a ring or field (section 2).

In section 3, we introduce *gradings* as morphism between fusion laws and group fusion laws. This will be an essential ingredient to make the connection between (axial) decomposition algebras and groups later on. We also explain how to construct such gradings for a given fusion law.

In section 4, we introduce *decomposition algebras*. These algebras axiomatize the essence of Griess algebras, Majorana algebras and axial algebras. We believe that this definition is the right approach to study all known algebras that are reminiscent of axial algebras. Moreover, it is the first definition in this context that allows for a suitable definition of a homomorphism and hence fits into a categorical framework. We explore this framework thoroughly in Appendix A.

In section 5, we explain how axial algebras fit into this framework by defining *axial decomposition algebras* and homomorphisms between axial decomposition algebras.

The important connection between decomposition algebras and groups is discussed in section 6, which is the longest section of the paper. We explain why the "obvious" connection (the *Miyamoto group*) is not functorial. However, we introduce a more universal connection (the *universal Miyamoto group*) which turns out to be functorial under some mild conditions. This is the subject of Proposition 6.9 and Theorem 6.12.

In section 7 and section 8, we present an important source of examples of decomposition algebras for a given (permutation) group. This is very closely related to *representation theory* and to the theory of *association schemes* via *Norton algebras*.

Acknowledgments. We thank the referees for their valuable and insightful comments. We also thank Jon Hall for his suggestions on how to improve the exposition of the paper.

Notation 1.1. We will use functional notation for our maps and morphisms, i.e., when $\varphi\colon A \to B$ is a map, we denote the image of an element a by $\varphi(a)$. Consequently, we will also denote conjugation of group elements on the left:

$$^{g}h := ghg^{-1}.$$

2. Fusion laws

In this section, we define (general) fusion laws. In contrast to previous definitions, these will no longer depend on a ring or a field.

Definition 2.1. A *fusion law*[1] is a pair $(X, *)$ where X is a set[2] and $*$ is a map from $X \times X$ to 2^X, where 2^X denotes the power set of X. A fusion law $(X, *)$ is called *symmetric* if $x * y = y * x$ for all $x, y \in X$.

[1] In earlier papers on axial algebras, this was referred to as "the fusion rules", leading to singular/plural problems. It has also been referred to as a "fusion table".

[2] The set X is often, but not always, a finite set.

Definition 2.2. Let $(X, *)$ be a fusion law and let $e \in X$.
- (i) We call e a *unit* if $e * x \subseteq \{x\}$ and $x * e \subseteq \{x\}$ for all $x \in X$.
- (ii) We call e *annihilating* if $e * x = \varnothing$ and $x * e = \varnothing$ for all $x \in X$.
- (iii) We call e *absorbing* if $e * x \subseteq \{e\}$ and $x * e \subseteq \{e\}$ for all $x \in X$.

Lemma 2.3. *Let $(X, *)$ be a fusion law. If $e, f \in X$ are units with $e \neq f$, then $e * f = \varnothing$.*

Proof. We have both $e * f \subseteq \{e\}$ and $e * f \subseteq \{f\}$. □

Example 2.4 (Jordan fusion law)**.** Consider the set $X = \{e, z, h\}$ with the symmetric fusion law

$*$	e	z	h
e	$\{e\}$	$\varnothing$	$\{h\}$
z	$\varnothing$	$\{z\}$	$\{h\}$
h	$\{h\}$	$\{h\}$	$\{e, z\}$

Here both e and z are units and accordingly $e * z = \varnothing$.

Example 2.5 (Ising fusion law)**.** Consider the set $X = \{e, z, q, t\}$ with the symmetric fusion law

$*$	e	z	q	t
e	$\{e\}$	$\varnothing$	$\{q\}$	$\{t\}$
z	$\varnothing$	$\{z\}$	$\{q\}$	$\{t\}$
q	$\{q\}$	$\{q\}$	$\{e, z\}$	$\{t\}$
t	$\{t\}$	$\{t\}$	$\{t\}$	$\{e, z, q\}$

Again, both e and z are units.

Remark 2.6. A fusion law $(X, *)$ can also be viewed as a map $\omega \colon X \times X \times X \to \{0, 1\}$, where we define $\omega(x, y, z) = 1 \iff z \in x * y$. As such, it is clear that there is an action of $\mathrm{Sym}(3)$ on the set of all fusion laws. It turns out that the Jordan fusion law and the Ising fusion law are invariant under this action.

Definition 2.7. Let $(X, *)$ and $(Y, *)$ be two fusion laws. A *morphism* from $(X, *)$ to $(Y, *)$ is a map $\xi \colon X \to Y$ such that

$$\xi(x_1 * x_2) \subseteq \xi(x_1) * \xi(x_2)$$

for all $x_1, x_2 \in X$, where we have denoted the obvious extension of ξ to a map $2^X \to 2^Y$ also by ξ.

This makes the set of all fusion laws into a category **Fus**.

Definition 2.8. Let $(X, *)$ and $(Y, *)$ be two fusion laws.
- (i) We define the *product* of $(X, *)$ and $(Y, *)$ to be the fusion law $(X \times Y, *)$ given by

$$(x_1, y_1) * (x_2, y_2) := \{(x, y) \mid x \in x_1 * x_2, y \in y_1 * y_2\}.$$

- (ii) We define the *union* of $(X, *)$ and $(Y, *)$ to be the fusion law $(X \cup Y, *)$, where $*$ extends the given fusion laws on X and Y and is defined by

$$x * y := \varnothing$$

for all $x \in X$ and all $y \in Y$.

Proposition 2.9. *The product and coproduct in the category* **Fus** *are given by the product and union of fusion laws, respectively, as defined in Definition 2.8.*

Proof. This follows easily from the definitions. Notice, in particular, that for given fusion laws $(X, *)$ and $(Y, *)$, the projection maps $X \times Y \to X$ and $X \times Y \to Y$ and the inclusion maps $X \to X \cup Y$ and $Y \to X \cup Y$ indeed induce morphisms in **Fus** as in Definition 2.7. □

An important class of fusion laws are the group fusion laws.

Definition 2.10. Let Γ be a group. Then the map

$$*\colon \Gamma \times \Gamma \to 2^{\Gamma} \colon (g, h) \mapsto \{gh\}$$

is a *group fusion law*. The identity element of Γ is the unique unit of the fusion law $(\Gamma, *)$.

Remark 2.11. The category **Grp** of groups is a *full* subcategory of **Fus**: if Γ and Δ are groups, then the fusion law morphisms from $(\Gamma, *)$ to $(\Delta, *)$ are precisely those arising from homomorphisms from Γ to Δ.

Two further examples of fusion laws arising in group theory and representation theory are given in the following examples.

Example 2.12 (Class fusion law)**.** Let G be a group with a finite number of conjugacy classes and let X be the set of those conjugacy classes. Then we can define a fusion law on X by declaring

$$E \in C * D \iff E \cap CD \neq \varnothing,$$

where CD is the setwise product of C and D inside G. The trivial conjugacy class $\{1\} \subseteq G$ is a unit for this fusion law. If G is a finite abelian group, this fusion law coincides with the group fusion law introduced in Definition 2.10.

Example 2.13 (Representation fusion law)**.** Let G be a finite group and let $X = \mathrm{Irr}(G)$ be its set of irreducible (complex) characters. Then we can define a fusion law on X by declaring

$$\chi \in \chi_1 * \chi_2 \iff \chi \text{ is a constituent of } \chi_1 \otimes \chi_2.$$

The trivial character is a unit for this fusion law.

3. Gradings

This section introduces the necessary preparations for the important connection between axial algebras and groups. On the level of fusion laws, this connection boils down to a morphism from a given fusion law to a group fusion law. We illustrate how to get the strongest possible connection by introducing the finest (abelian) grading of a fusion law.

Definition 3.1. (i) Let $(X, *)$ be a fusion law and let $(\Gamma, *)$ be a group fusion law. A Γ-*grading* of $(X, *)$ is a morphism $\xi\colon (X, *) \to (\Gamma, *)$. We call the grading *abelian* if Γ is an abelian group and we call it *adequate* if $\xi(X)$ generates Γ.

(ii) Every fusion law admits a Γ-grading where Γ is the trivial group; we call this the *trivial* grading.

(iii) Let $(X, *)$ be a fusion law. We say that a Γ-grading ξ of $(X, *)$ is a *finest grading* of $(X, *)$ if every grading of $(X, *)$ factors uniquely through $(\Gamma, *)$, in other words, if for each Λ-grading ζ of $(X, *)$, there is a unique group homomorphism $\rho\colon \Gamma \to \Lambda$ such that $\zeta = \rho \circ \xi$. (In categorical terms, this can be rephrased as the fact that ξ is an initial object in the category of gradings of $(X, *)$.)

Similarly, we say that an abelian Γ-grading ξ of $(X, *)$ is a *finest abelian grading* of $(X, *)$ if every abelian grading of $(X, *)$ factors uniquely through $(\Gamma, *)$.

Proposition 3.2. *Every fusion law $(X, *)$ admits a unique finest grading, given by the group with presentation*

$$\Gamma_X := \langle \gamma_x, x \in X \mid \gamma_x \gamma_y = \gamma_z \text{ whenever } z \in x * y \rangle,$$

with grading map $\xi\colon (X, *) \to (\Gamma_X, *)\colon x \mapsto \gamma_x$. Similarly, there is a unique finest abelian grading, given by the abelianization $\Gamma_X/[\Gamma_X, \Gamma_X]$ of Γ_X. Both gradings are adequate.

Proof. In order to verify that the map $\xi\colon (X, *) \to (\Gamma_X, *)\colon x \mapsto \gamma_x$ is a morphism of fusion laws, we have to check that $\xi(z) \in \xi(x) * \xi(y)$ for all $z \in x * y$. This is clear from the definition of Γ_X, since $\xi(z) = \gamma_z$ and $\xi(x) * \xi(y) = \{\gamma_x \gamma_y\}$. Clearly, ξ is then an adequate grading since Γ_X is generated by the elements γ_x.

Assume now that $\zeta\colon (X, *) \to (\Lambda, *)$ is another grading of $(X, *)$. If $x, y, z \in X$ satisfy $z \in x * y$, then $\zeta(z) \in \zeta(x) * \zeta(y) = \{\zeta(x)\zeta(y)\}$, so the elements $\zeta(x)$ satisfy the defining relations of the generators γ_x in the presentation for Γ_X. This implies that the map $\rho\colon \Gamma_X \to \Lambda\colon \gamma_x \mapsto \zeta(x)$ is a well defined group homomorphism, with $\zeta = \rho \circ \xi$. Since ξ is adequate, the identity $\zeta = \rho \circ \xi$ also uniquely determines the group homomorphism ρ.

The proof of the remaining statement is similar. □

Remark 3.3. There is a lot of "collapsing" in the group Γ_X:

(a) If $y \in x * y$ for some $y \in X$, then $\gamma_x = 1$ in Γ_X. In particular, $\gamma_x = 1$ for each non-annihilating unit $x \in X$.
(b) All γ_z, where z runs through some fixed set $x * y$, are equal to each other in Γ_X.
(c) If z belongs to $x * y$ and to $x * y'$, then $\gamma_y = \gamma_{y'}$. Similarly, if z belongs to $x * y$ and to $x' * y$, then $\gamma_x = \gamma_{x'}$.

From this it is clear that Γ_X is trivial for most fusion laws $(X, *)$, i.e., they only admit the trivial grading. We call a fusion law $(X, *)$ *graded* if $\Gamma_X \neq 1$ and *ungraded* otherwise. It will turn out that graded fusion laws are more interesting for our purposes.

Example 3.4. The Jordan fusion law in Example 2.4 is $\mathbb{Z}/2\mathbb{Z}$-graded. Indeed, the map $\xi\colon X \to \mathbb{Z}/2\mathbb{Z}$ mapping e and z to 0 and h to 1 is a fusion law morphism. Notice that this is the finest grading of the Jordan fusion law.

Similarly, the Ising fusion law in Example 2.5 admits a $\mathbb{Z}/2\mathbb{Z}$-grading: the map $\xi\colon X \to \mathbb{Z}/2\mathbb{Z}$ mapping e, z and q to 0 and t to 1 is a fusion law morphism. Again, this is the finest grading of the Ising fusion law.

In the remainder of this section, we describe the finest grading of two special types of fusion laws: class fusion laws and representation fusion laws.

The class fusion law of a group G was introduced in Example 2.12. For $g \in G$, let $\bar{g}$ denote the image of g in $G/[G, G]$.

Proposition 3.5. *Let $(X, *)$ be the class fusion law of a group G. Then the finest grading of $(X, *)$ is given by the group $\Gamma = G/[G, G]$ with grading map $X \to \Gamma\colon {}^G g \mapsto \overline{g}$.*

Proof. By definition, the finest grading of $(X, *)$ is the group

$$\Gamma_X := \langle \gamma_C, C \in X \mid \gamma_C \gamma_D = \gamma_E \text{ whenever } CD \cap E \neq \varnothing \rangle.$$

Consider the map $\varphi\colon G \to \Gamma_X\colon g \mapsto \gamma_{({}^G g)}$ and notice that φ is a group morphism, precisely by the defining relations of Γ_X. It is clearly surjective; moreover, $\varphi({}^g h) = \varphi(h)$ for all $g, h \in G$. It follows that for each commutator $[g, h] = ghg^{-1}h^{-1}$, we have $\varphi([g, h]) = \varphi({}^g h)\varphi(h)^{-1} = 1$; hence $[G, G] \leq \ker \varphi$. Hence φ induces a group epimorphism $\tilde{\varphi}\colon \Gamma \to \Gamma_X$.

Finally, the map $\Gamma_X \to \Gamma\colon \gamma_{({}^G g)} \to g[G, G]$ is well defined because it kills each relator of Γ_X, and this map provides an inverse of $\tilde{\varphi}$, showing that it is an isomorphism from Γ_X to Γ. □

Recall the definition of the representation fusion law from Example 2.13.

Proposition 3.6. *Let G be a finite group and let $(X,*)$ be the representation fusion law of G. Then the finest grading of $(X,*)$ is given by $\Gamma_X = Z(G)^* = \operatorname{Irr}(Z(G))$ with grading map $X \to \operatorname{Irr}(Z(G))\colon \chi \mapsto \frac{\chi_{Z(G)}}{\chi(1)}$.*

Proof. [Proof[3]] Consider an arbitrary adequate grading $f\colon X \to \Gamma$ and define

$$K = \{\chi \in \operatorname{Irr}(G) \mid f(\chi) = 1\}.$$

Let $H = \bigcap_{\chi \in K} \ker \chi$. If $\chi \in K$ then it is clear that $H \leq \ker \chi$; we aim to show the opposite inclusion. Consider $\theta = \sum_{\chi \in K} \chi$, which may be considered as a character of G/H. Since θ is faithful as a character of G/H, by the Burnside–Brauer theorem every irreducible character of G/H is a constituent of some power of θ. Now since f is trivial on each constituent of θ, it also is trivial on all irreducible characters of G/H. Thus if $H \leq \ker \chi$ then $\chi \in K$. We have now established that $K = \{\chi \in \operatorname{Irr}(G) \mid H \leq \ker \chi\}$.

Note that $f(\bar{\chi}) = f(\chi)^{-1}$. Indeed, $\mathbb{1}_G$ is a constituent of $\chi\bar{\chi}$; that is, $\mathbb{1}_G \in \chi * \bar{\chi}$. This means that $f(\chi)f(\bar{\chi}) = f(\mathbb{1}_G) = 1$.

Now let $\psi \in \operatorname{Irr}(H)$ and let χ and η be constituents of the induced character ψ^G, so that ψ is a constituent of the restrictions χ_H and η_H by Frobenius reciprocity. Thus $0 < \langle \eta_H, \chi_H \rangle = \langle \mathbb{1}_H, (\chi\overline{\eta})_H \rangle$ (where $\langle -,- \rangle$ represents the inner product of class functions) and hence $\mathbb{1}_H$ is a constituent of $(\chi\overline{\eta})_H$. Since $H \trianglelefteq G$, a corollary of Clifford's theorem now implies that $\chi\overline{\eta}$ has a constituent $\theta \in \operatorname{Irr}(G)$ with $H \leq \ker\theta$ (see for example [Isa94]*Corollary 6.7). Hence $f(\chi)f(\eta)^{-1} = f(\chi\overline{\eta}) = f(\theta) = 1$. That is, $f(\chi) = f(\eta)$. Thus, we obtain a well-defined map $f'\colon \operatorname{Irr}(H) \to \Gamma$ by setting $f'(\psi) = f(\chi)$ for any constituent χ of ψ^G.

Next, we show that H is in the center of G, so let us assume that there is some non-central $x \in H$. As x is not central, the column orthogonality relations imply that there must be a character $\chi \in \operatorname{Irr}(G)$ such that $|\chi(x)| \leq \chi(1)$ and, therefore, there is a constituent θ of $\chi\overline{\chi}$ with $\theta(x) \neq \theta(1)$. On the other hand, $f(\theta) = f(\chi)f(\overline{\chi}) = 1$, yielding $\theta \in K$. This means that $H \leq \ker\theta$ and so $\theta(x) = \theta(1)$; a contradiction.

Since H is central, the map $X \to \operatorname{Irr}(H)\colon \chi \mapsto \frac{\chi_H}{\chi(1)}$ is defined and f is the composition of this map and f'. Clearly, the map $X \to \operatorname{Irr}(H)$ factors through the similar map $X \to \operatorname{Irr}(Z(G))$, and so the claim of the proposition holds. ☐

Remark 3.7. (i) It is immediate from the definition that the finest grading of the union of fusion laws $(X,*)$ and $(Y,*)$ is the free product of Γ_X and Γ_Y with the obvious grading map.

(ii) The similar question about the finest grading of the product $(X \times Y, *)$ is more difficult. It is easy to see that there is a grading of $(X \times Y, *)$ by the group $\Gamma_X \times \Gamma_Y$. However, it is equally easy to find examples where this is not the finest grading. For instance, if $(X,*)$ is an empty fusion law (i.e., $x_1 * x_2 = \varnothing$ for all $x_1, x_2 \in X$) and $(Y,*)$ is any fusion law, then the product $(X \times Y, *)$ is again an empty fusion law, but the finest grading of an empty fusion law is always a free group.

4. Decomposition algebras

We are now ready to introduce decomposition algebras. We believe that they provide the right axiomatic framework to study all algebras reminiscent of axial algebras. It is the first definition of such algebras that allows for an interesting definition of homomorphisms. For each choice

[3] Thanks to David Craven and Frieder Ladisch for providing the central argument in this proof. As Frieder Ladisch pointed out to us, this result also follows from [GN08]*Example 3.2 and Corollary 3.7.

[GN08] Gelaki and Nikshych, *Nilpotent fusion categories*, Adv. Math. **217** (2008), no. 3, 1053–1071

[Isa94] Isaacs, *Character theory of finite groups*, Dover Publications, Inc., New York, 1994

of a base ring and a fusion law, this will give rise to a corresponding *category of decomposition algebras*. We refer to Appendix A for further categorical properties.

Definition 4.1. Let R be a commutative ring and let $\Phi = (X, *)$ be a fusion law.

(i) A Φ-*decomposition* of an R-algebra A (not assumed to be commutative, associative or unital) is a direct sum decomposition $A = \bigoplus_{x \in X} A_x$ (as R-modules) such that $A_x A_y \subseteq A_{x*y}$ for all $x, y \in X$, where $A_Y := \bigoplus_{y \in Y} A_y$ for all $Y \subseteq X$.

(ii) A Φ-*decomposition algebra* is a triple $(A, \mathcal{I}, \Omega)$ where A is an R-algebra, $\mathcal{I}$ is an index set and Ω is a tuple[4] of Φ-decompositions of A indexed by $\mathcal{I}$. We will usually write the corresponding decompositions as $A = \bigoplus_{x \in X} A^i_x$, so

$$\Omega = \big((A^i_x)_{x \in X} \mid i \in \mathcal{I} \big);$$

we sometimes use the shorthand notation $\Omega[i] := (A^i_x)_{x \in X}$. Notice that we do not require the decompositions to be distinct.

[4]Formally, we could define Ω as a set and define this "tuple" as a map from $\mathcal{I}$ to Ω, but we will not do so in order not to make our notation unnecessarily complicated.

We will often omit the explicit reference to Φ if it is clear from the context and simply talk about decompositions and decomposition algebras.

Example 4.2. Consider the following fusion law $(X, *)$ on $X = \{e, z\}$:

$*$	e	z
e	$\{e\}$	$\emptyset$
z	$\emptyset$	$\{z\}$

Let A be any *commutative associative* algebra over a commutative ring R. Let $\{a_i \mid i \in \mathcal{I}\} \subseteq A$ be any collection of idempotents in A, indexed by some set $\mathcal{I}$. For each $i \in \mathcal{I}$, the algebra A decomposes as $A = a_i A \oplus (1 - a_i)A$. Write $A^i_e := a_i A$ and $A^i_z := (1 - a_i)A$. Then each decomposition $A = A^i_e \oplus A^i_z$ is indeed an $(X, *)$-decomposition. If we write Ω for the $\mathcal{I}$-tuple of all those decompositions, then $(A, \mathcal{I}, \Omega)$ is a decomposition algebra.

Example 4.3. Consider the Jordan fusion law $(X, *)$ from Example 2.4. Let J be any *Jordan* algebra over a commutative ring R. Let $\{a_i \mid i \in \mathcal{I}\} \subseteq J$ be any collection of idempotents in J, indexed by some set $\mathcal{I}$. For each $i \in \mathcal{I}$, the algebra J admits a *Peirce decomposition* into the *Peirce subspaces* with respect to the idempotent a_i (see, e.g., [Jac68, Chapter III]):

$$J = J^i_0 \oplus J^i_1 \oplus J^i_{1/2}.$$

[Jac68] Jacobson, *Structure and representations of Jordan algebras*, American Mathematical Society Colloquium Publications, Vol. XXXIX, American Mathematical Society, Providence, R.I., 1968

By [Jac68, Chapter III, §1, Lemma 1], each of those decompositions is indeed an $(X, *)$-decomposition (where e corresponds to 1, z to 0 and h to 1/2). If we write Ω for the $\mathcal{I}$-tuple of all those decompositions, then $(A, \mathcal{I}, \Omega)$ is a decomposition algebra.

Remark 4.4. Let $\Phi = (X, *)$ be a fusion law and let $(A, \mathcal{I}, \Omega)$ be a Φ-decomposition algebra. If $e \in X$ is annihilating (see Definition 2.2), then each subspace A^i_e is annihilating for the algebra A, in the sense that $A^i_e \cdot A = 0$. Similarly, if $e \in X$ is absorbing (see Definition 2.2), then each A^i_e is an ideal: $A^i_e \cdot A \subseteq A^i_e$.

The decomposition algebras with respect to a fixed fusion law form a nice category.

Definition 4.5. Let R be a commutative ring and let $\Phi = (X, *)$ be a fusion law. We define a category Φ-$\mathbf{Dec}_R$ having as objects the Φ-decomposition algebras over R. If $(A, \mathcal{I}, \Omega_A)$ and $(B, \mathcal{J}, \Omega_B)$ are two objects, with

$$\Omega_A = \big((A^i_x)_{x \in X} \mid i \in \mathcal{I} \big), \qquad \Omega_B = \big((B^j_x)_{x \in X} \mid j \in \mathcal{J} \big),$$

then the *morphisms* between $(A, \mathcal{I}, \Omega_A)$ and $(B, \mathcal{J}, \Omega_B)$ are defined to be pairs (φ, ψ) where $\varphi: A \to B$ is an R-algebra morphism and $\psi: \mathcal{I} \to \mathcal{J}$ is a map (of sets) such that

$$\varphi(A^i_x) \subseteq B^{\psi(i)}_x$$

for all $x \in X$ and all $i \in \mathcal{I}$.

Proposition 4.6. *If $\xi: (X, *) \to (Y, *)$ is a fusion law morphism and $(A, \mathcal{I}, \Omega)$ is an $(X, *)$-decomposition algebra, then A can also be viewed as a $(Y, *)$-decomposition algebra $(A, \mathcal{I}, \Sigma)$ by declaring*

$$A^i_y := A^i_{\xi^{-1}(y)} = \bigoplus_{x \in \xi^{-1}(y)} A^i_x$$

for each $i \in \mathcal{I}$ and each $y \in Y$. This induces a functor

$$F_\xi: (X, *)\text{-}\mathbf{Dec}_R \to (Y, *)\text{-}\mathbf{Dec}_R.$$

Proof. We have to verify that for all $y, z \in Y$, we have $A^i_y A^i_z \subseteq A^i_{y*z}$. By the definition of a fusion law morphism, we have

$$\xi^{-1}(y) * \xi^{-1}(z) \subseteq \xi^{-1}(y * z),$$

and hence indeed

$$\begin{aligned} A^i_y A^i_z &= A^i_{\xi^{-1}(y)} A^i_{\xi^{-1}(z)} \\ &\subseteq A^i_{\xi^{-1}(y) * \xi^{-1}(z)} \\ &\subseteq A^i_{\xi^{-1}(y*z)} = A^i_{y*z}, \end{aligned}$$

proving the proposition. □

In Appendix A, we study the category Φ-$\mathbf{Dec}_R$ in some more detail.

5. Axial decomposition algebras

In this section, we explain how axial algebras fit into the framework of decomposition algebras.

Definition 5.1. Let $\Phi = (X, *)$ be a fusion law with a distinguished unit $e \in X$. For each $x \in X$, let $\lambda_x \in R$. A Φ-decomposition algebra $(A, \mathcal{I}, \Omega)$ will be called *left-axial* (with *parameters* λ_x) if for each $i \in \mathcal{I}$, there is some non-zero $a_i \in A^i_e$ (called a *left axis*) such that:

(1) $$a_i \cdot b = \lambda_x b \quad \text{for all } x \in X \text{ and for all } b \in A^i_x.$$

Similarly, $(A, \mathcal{I}, \Omega)$ is a *right-axial* decomposition algebra (with *parameters* λ_x) if for each $i \in \mathcal{I}$, there is some non-zero $a_i \in A^i_e$ (called a *right axis*) such that:

(2) $$b \cdot a_i = \lambda_x b \quad \text{for all } x \in X \text{ and for all } b \in A^i_x.$$

Of course, if A is commutative, then we drop the prefix "left" or "right" and simply talk about axial decomposition algebras. We call a (left- or right-)axial decomposition algebra *primitive* if $A^i_e = Ra_i$ for each $i \in \mathcal{I}$.

Remark 5.2. Recall from [HRS15b] that an *axial algebra* is a commutative algebra A generated by a set E of idempotents (called *axes*), such that for each axis $c \in E$, the left multiplication operator $\mathrm{ad}_c: A \to A: x \mapsto cx$ is semi-simple and its eigenspaces multiply according to a given fusion law $\Phi = (X, *)$ with $X \subseteq R$.

[HRS15b] Hall, Rehren, and Shpectorov, *Universal axial algebras and a theorem of Sakuma*, J. Algebra **421** (2015), 394–424

Every axial algebra is an axial decomposition algebra. Indeed, if (A, E) is an axial algebra, then for each $c \in E$, there is a corresponding decomposition $A = \bigoplus_{x \in X} A^c_x$, so certainly (A, E, Ω) with $\Omega = \{(A^c_x)_{x \in X} \mid c \in E\}$ is a decomposition algebra. It is indeed axial, with $a_c = c$ for each $c \in E \subseteq A$ and $\lambda_x = x$ for each $x \in X \subseteq R$.

On the other hand, axial decomposition algebras are more general objects than axial algebras, in four ways:

- The elements $a_c \in A$ are not required to be idempotents. If the corresponding parameter $\lambda_e \neq 0$ is a unit in R (for example when R is a field), then we can rescale a_c to an idempotent. If $\lambda_e = 0$, then $a_c^2 = 0$, i.e., a_c is nilpotent.
- The algebra A is not assumed to be generated by the axes.
- By distinguishing between $x \in X$ and $\lambda_x \in R$, we allow the possibility that some of the $\lambda_x \in R$ coincide.
- The algebra A is not assumed to be commutative.

We now make the class of (left) axial decomposition algebras into a category.

Definition 5.3. Let $\Phi = (X, *)$ be a fusion law with a distinguished unit $e \in X$ and let $\lambda\colon X \to R\colon x \mapsto \lambda_x$ be an arbitrary map, called the *evaluation map*. We define a category (Φ, λ)-$\mathbf{AxDec}_R$ with as objects the axial Φ-decomposition algebras together with the collection of left axes, for the choice of parameters λ_x given by the evaluation map. In other words, the objects are quadruples $(A, \mathcal{I}, \Omega, \alpha)$, where $(A, \mathcal{I}, \Omega)$ is a Φ-decomposition algebra and $\alpha\colon \mathcal{I} \to A\colon i \mapsto a_i$ is a map such that $a_i \in A_e^i$ and (1) holds.

The morphisms in this category are the morphisms $(\varphi, \psi)\colon (A, \mathcal{I}, \Omega_A, \alpha) \to (B, \mathcal{J}, \Omega_B, \beta)$ of decomposition algebras such that $\varphi \circ \alpha = \beta \circ \psi$, i.e., φ maps each axis a_i to the corresponding axis $b_{\psi(i)}$.

6. The (universal) Miyamoto groups

Let Γ be a finite group fusion law. To each Γ-decomposition algebra $(A, \mathcal{I}, \Omega)$, we will associate a subgroup of the automorphism group of A, called the Miyamoto group of $(A, \mathcal{I}, \Omega)$. We will also construct a cover of this group, which we call the universal Miyamoto group and which has nicer functorial properties than the Miyamoto group itself.

We will, at the same time, construct subgroups of these Miyamoto groups, one for each subgroup of the character group.

Definition 6.1. Let $R^\times$ be the group of invertible elements of the base ring R. An *R-character* of Γ is a group homomorphism $\chi\colon \Gamma \to R^\times$. The *$R$-character group* of Γ is the group $\mathcal{X}_R(\Gamma)$ consisting of all R-characters of Γ, with group operation induced by multiplication in $R^\times$. When the base ring R is clear from the context, we will sometimes omit it and simply talk about characters and the character group.

Notice that depending on R, the group $\mathcal{X}_R(\Gamma)$ might be infinite even if Γ is finite.

Definition 6.2. Let $(A, \mathcal{I}, \Omega)$ be a Γ-decomposition algebra.

(i) Let $\chi \in \mathcal{X}_R(\Gamma)$. For each decomposition $(A_g^i)_{g \in \Gamma} \in \Omega$, we define a linear map

$$\tau_{i,\chi}\colon A \to A\colon a \mapsto \chi(g)a \quad \text{for all } a \in A_g^i;$$

we call this a *Miyamoto map*. It follows immediately from the definitions that each $\tau_{i,\chi}$ is an automorphism of the R-algebra A. Notice that each $\tau_{i,\chi}$ has finite order (dividing the order of χ in $\mathcal{X}_R(\Gamma)$).

(ii) Let $\mathcal{Y}$ be any subgroup of the character group $\mathcal{X}_R(\Gamma)$. We then define the *Miyamoto group* with respect to $\mathcal{Y}$ as

$$\mathrm{Miy}_{\mathcal{Y}}(A, \mathcal{I}, \Omega) := \langle \tau_{i,\chi} \mid i \in \mathcal{I}, \chi \in \mathcal{Y} \rangle \leq \mathrm{Aut}(A).$$

Two important special cases get their own notation:

$$\operatorname{Miy}(A,\mathcal{I},\Omega) := \operatorname{Miy}_{\mathcal{X}_R(\Gamma)}(A,\mathcal{I},\Omega);$$
$$\operatorname{Miy}_\chi(A,\mathcal{I},\Omega) := \operatorname{Miy}_{\langle\chi\rangle}(A,\mathcal{I},\Omega) \quad \text{for a given character } \chi \in \mathcal{X}_R(G).$$

(iii) We call $(A,\mathcal{I},\Omega)$ *Miyamoto-closed* with respect to $\mathcal{Y}$ if the set Ω is invariant under the Miyamoto group with respect to $\mathcal{Y}$. That is for each $i \in \mathcal{I}$ and each $\chi \in \mathcal{Y}$, there is a permutation[5] $\pi_{i,\chi}$ of $\mathcal{I}$ such that $\tau_{i,\chi}$ maps each decomposition $(A^j_g)_{g\in\Gamma} \in \Omega$ to the decomposition $(A^{\pi_{i,\chi}(j)}_g)_{g\in\Gamma} \in \Omega$. Notice that in this case, each pair $(\tau_{i,\chi},\pi_{i,\chi})$ is an automorphism of $(A,\mathcal{I},\Omega)$ in the category Γ-$\mathbf{Dec}_R$. In particular, the conjugate of a Miyamoto map by a Miyamoto map is again a Miyamoto map.

[5] In the situation where some of the decompositions $(A^j_g)_{g\in\Gamma} \in \Omega$ coincide, there might be some freedom in the choice of the permutation $\pi_{i,\chi}$, but this choice will be irrelevant for us.

Example 6.3. The simplest non-trivial example is the case where $\Gamma = \mathbb{Z}/2\mathbb{Z}$ and $\mathcal{Y} = \{1,\chi\}$ where χ maps the non-trivial element of Γ to $-1 \in R$ (assuming that $-1 \neq 1$ in R). In the case of axial algebras, we recover the definition of the Miyamoto group as in [DMVC18]*Definition 2.5.

[DMVC18] De Medts and Van Couwenberghe, *Modules over axial algebras*, Algebr. Represent. Theory (2018), To appear. doi: 10.1007/s10468-018-9844-y

The Miyamoto group is interesting—it is a subgroup of the automorphism group of the algebra—but is not so easy to control (cf. Example 6.13 below). It is useful to construct a cover of this group, which we call the *universal Miyamoto group*.

Definition 6.4. We keep the notations from Definition 6.2 and assume that $(A,\mathcal{I},\Omega)$ is Miyamoto-closed with respect to $\mathcal{Y}$. Recall our convention from Notation 1.1. We define the *universal Miyamoto group* with respect to $\mathcal{Y}$ as the group given by the following presentation. For each $i \in I$, we let $\mathcal{Y}_i$ be a copy of the group $\mathcal{Y}$ and we denote its elements by

$$\mathcal{Y}_i = \{t_{i,\chi} \mid \chi \in \mathcal{Y}\}.$$

For each $a = t_{i,\chi} \in \mathcal{Y}_i$, we write $\overline{a}$ for the corresponding Miyamoto map $\tau_{i,\chi} \in \operatorname{Miy}(A,\mathcal{I},\Omega)$. Notice that for each $i \in \mathcal{I}$, the group

$$\overline{\mathcal{Y}_i} := \{\overline{a} \mid a \in \mathcal{Y}_i\} = \{\tau_{i,\chi} \mid \chi \in \mathcal{Y}\}$$

is an *abelian* subgroup of $\operatorname{Miy}_{\mathcal{Y}}(A,\mathcal{I},\Omega)$.

We will define the universal Miyamoto group $\widehat{\operatorname{Miy}}_{\mathcal{Y}}(A,\mathcal{I},\Omega)$ as a quotient of the free product $\ast_{i\in\mathcal{I}}\mathcal{Y}_i$ by conjugation relations between the groups $\mathcal{Y}_i$ that exist "globally" between the corresponding groups $\overline{\mathcal{Y}_i}$ in $\operatorname{Miy}(A,\mathcal{I},\Omega)$. More precisely, let $\mathcal{U} := \bigcup_{i\in\mathcal{I}}\mathcal{Y}_i$; for each $a \in \mathcal{U}$, we consider the set

$$R_{\overline{a}} := \{(j,k) \in \mathcal{I}\times\mathcal{I} \mid {}^{\overline{a}}\tau_{j,\chi} = \tau_{k,\chi} \text{ for all } \chi \in \mathcal{Y}\}. \tag{3}$$

We then let

$$\widehat{\operatorname{Miy}}_{\mathcal{Y}}(A,\mathcal{I},\Omega) := \left\langle \mathop{\ast}_{i\in\mathcal{I}} \mathcal{Y}_i \;\middle|\; {}^{a}t_{j,\chi} = t_{k,\chi} \text{ for all } a \in \mathcal{U}, \text{ all } (j,k) \in R_{\overline{a}} \text{ and all } \chi \in \mathcal{Y} \right\rangle.$$

Remark 6.5. The reader might wonder why we only consider conjugation relations that exist globally and do not define the universal Miyamoto group as the group

$$\left\langle \mathop{\ast}_{i\in\mathcal{I}} \mathcal{Y}_i \;\middle|\; {}^{a}b = c \text{ for all } a,b,c \in \mathcal{U} \text{ satisfying } {}^{\overline{a}}\overline{b} = \overline{c} \right\rangle.$$

instead. The problem with this definition is that some conjugation relations might hold "by coincidence" and we do not want to transfer those to the universal Miyamoto group. For instance, Theorem 6.12 below would become false with this seemingly easier definition.

On the other hand, since $(A, \mathcal{I}, \Omega)$ is Miyamoto-closed with respect to $\mathcal{Y}$, we always have many conjugation relations at our disposal.

Lemma 6.6. *Let $i, j \in \mathcal{I}$.*

(i) *For each $\chi, \chi' \in \mathcal{Y}$, the relation*

$$^{t_{i,\chi}} t_{j,\chi'} = t_{\pi_{i,\chi}(j),\chi'}$$

holds in $\widehat{\mathrm{Miy}}_{\mathcal{Y}}(A, \mathcal{I}, \Omega)$.

(ii) *If $\tau_{i,\chi} = \tau_{j,\chi}$ for all $\chi \in \mathcal{Y}$, then also $t_{i,\chi} = t_{j,\chi}$ for all $\chi \in \mathcal{Y}$.*

Proof.

(i) Let $a = t_{i,\chi}$ for some $\chi \in \mathcal{Y}$. Since $(A, \mathcal{I}, \Omega)$ is Miyamoto-closed with respect to $\mathcal{Y}$, we have $^{\tau_{i,\chi}} \tau_{j,\chi'} = \tau_{\pi_{i,\chi}(j),\chi'}$ for all $\chi' \in \mathcal{Y}$ and therefore $(j, \pi_{i,\chi}(j)) \in R_{\overline{a}}$. It follows that all relations of the form

$$^{t_{i,\chi}} t_{j,\chi'} = t_{\pi_{i,\chi}(j),\chi'}$$

hold in $\widehat{\mathrm{Miy}}_{\mathcal{Y}}(A, \mathcal{I}, \Omega)$.

(ii) Let $a = t_{i,\chi}$ for some $\chi \in \mathcal{Y}$. Recall that $\overline{\mathcal{Y}_i}$ is abelian, hence $\tau_{i,\chi}$ commutes with $\tau_{i,\chi'}$ for all $\chi' \in \mathcal{Y}$. Since $\tau_{i,\chi'} = \tau_{j,\chi'}$, it follows that $(i, j) \in R_{\overline{a}}$. Therefore, the relations

$$^{t_{i,\chi}} t_{i,\chi'} = t_{j,\chi'}$$

hold in $\widehat{\mathrm{Miy}}_{\mathcal{Y}}(A, \mathcal{I}, \Omega)$. Since $t_{i,\chi}$ and $t_{i,\chi'}$ both belong to the abelian group $\mathcal{Y}_i \leq \widehat{\mathrm{Miy}}_{\mathcal{Y}}(A, \mathcal{I}, \Omega)$, we conclude that the relation $t_{i,\chi'} = t_{j,\chi'}$ holds in $\widehat{\mathrm{Miy}}_{\mathcal{Y}}(A, \mathcal{I}, \Omega)$. □

Proposition 6.7. *Let $\mathcal{Y} \leq \mathcal{X}_R(\Gamma)$ and let $(A, \mathcal{I}, \Omega)$ be Miyamoto-closed with respect to $\mathcal{Y}$. Then $\widehat{\mathrm{Miy}}_{\mathcal{Y}}(A, \mathcal{I}, \Omega)$ is a central extension of $\mathrm{Miy}_{\mathcal{Y}}(A, \mathcal{I}, \Omega)$.*

Proof. Let $\widehat{G} := \widehat{\mathrm{Miy}}_{\mathcal{Y}}(A, \mathcal{I}, \Omega)$, $G := \mathrm{Miy}_{\mathcal{Y}}(A, \mathcal{I}, \Omega)$ and $\mathcal{U} := \bigcup_{i \in \mathcal{I}} \mathcal{Y}_i \subseteq \widehat{G}$; then $\widehat{G} = \langle \mathcal{U} \rangle$. It is immediately clear from the definition of $\widehat{\mathrm{Miy}}_{\mathcal{Y}}(A, \mathcal{I}, \Omega)$ that the map $\mathcal{U} \to G \colon a \mapsto \overline{a}$ extends to an epimorphism $\Phi \colon \widehat{G} \to G$; it remains to show that $\ker \Phi$ is central.

Let $z \in \ker \Phi$ be arbitrary; as each generator $a \in \mathcal{U}$ has finite order, we can write $z = a_m \cdots a_1$ with $a_i \in \mathcal{U}$. We have to show that $^z b = b$ for each $b = t_{j,\chi'} \in \mathcal{U}$. Fix such an element $b \in \mathcal{U}$. For each $k \in \{0, \dots, m\}$, we write

$$b_k := {}^{a_k \cdots a_1} b \in \widehat{G} \quad \text{and}$$
$$c_k := {}^{\overline{a}_k \cdots \overline{a}_1} \overline{b} \in G.$$

By repeatedly applying Lemma 6.6(i), we see that each b_k is again of the form $t_{j_k,\chi'}$ for some $j_k \in \mathcal{I}$ (which only depends on z and j but not on χ') and that $c_k = \overline{b_k}$ for each k.

In particular, $\overline{b_m} = c_m = {}^{\Phi(z)}\overline{b} = \overline{b}$ with $b = t_{j,\chi'}$ and $b_m = t_{j_m,\chi'}$. Hence $\tau_{j_m,\chi'} = \tau_{j,\chi'}$. Because this holds for all $\chi' \in \mathcal{Y}$, Lemma 6.6(ii) now implies that $t_{j_m,\chi'} = t_{j,\chi'}$ for all χ'. Varying $j \in \mathcal{I}$ finishes the proof. □

For *surjective* morphisms between decomposition algebras, both $\mathrm{Miy}_{\mathcal{Y}}$ and $\widehat{\mathrm{Miy}}_{\mathcal{Y}}$ are functorial. The following easy lemma is the key point.

Lemma 6.8. *Let (φ, ψ) be a morphism between two Γ-decomposition algebras $(A, \mathcal{I}, \Omega_A)$ and $(B, \mathcal{J}, \Omega_B)$. Then for each $i \in \mathcal{I}$ and $\chi \in \mathcal{X}_R(\Gamma)$, we have $\varphi \circ \tau_{i,\chi} = \tau_{\psi(i),\chi} \circ \varphi$.*

Proof. Let $a \in A^i_g$ for some $g \in \Gamma$. Then on the one hand, $\varphi(\tau_{i,\chi}(a)) = \varphi(\chi(g)a) = \chi(g)\varphi(a)$, while on the other hand, $\varphi(a) \in B^{\psi(i)}_g$ and hence $\tau_{\psi(i),\chi}(\varphi(a)) = \chi(g)\varphi(a)$ as well. Since $A = \bigoplus_{g \in \Gamma} A^i_g$, the result follows. □

Proposition 6.9. *Let $\mathcal{Y} \leq \mathcal{X}_R(\Gamma)$. Let (φ, ψ) be a morphism between two Γ-decomposition algebras $(A, \mathcal{I}, \Omega_A)$ and $(B, \mathcal{J}, \Omega_B)$. Assume that φ is surjective. Then:*

(i) *There is a corresponding morphism $\theta\colon \mathrm{Miy}_{\mathcal{Y}}(A, \mathcal{I}, \Omega_A) \to \mathrm{Miy}_{\mathcal{Y}}(B, \mathcal{J}, \Omega_B)$ mapping each generator $\tau_{i,\chi}$ of $\mathrm{Miy}_{\mathcal{Y}}(A, \mathcal{I}, \Omega_A)$ to the corresponding generator $\tau_{\psi(i),\chi}$ of $\mathrm{Miy}_{\mathcal{Y}}(B, \mathcal{J}, \Omega_B)$.*

(ii) *There is a corresponding morphism $\widehat{\theta}\colon \widehat{\mathrm{Miy}}_{\mathcal{Y}}(A, \mathcal{I}, \Omega_A) \to \widehat{\mathrm{Miy}}_{\mathcal{Y}}(B, \mathcal{J}, \Omega_B)$ mapping each generator $t_{i,\chi}$ of $\widehat{\mathrm{Miy}}_{\mathcal{Y}}(A, \mathcal{I}, \Omega_A)$ to the corresponding generator $t_{\psi(i),\chi}$ of $\widehat{\mathrm{Miy}}_{\mathcal{Y}}(B, \mathcal{J}, \Omega_B)$.*

Proof.

(i) It suffices to verify that if the $\tau_{i,\chi}$ satisfy some relation

$$\tau_{i_1,\chi_1} \cdots \tau_{i_\ell,\chi_\ell} = 1$$

inside $\mathrm{Aut}(A)$, then also

$$\tau_{\psi(i_1),\chi_1} \cdots \tau_{\psi(i_\ell),\chi_\ell} = 1$$

inside $\mathrm{Aut}(B)$. This follows immediately from Lemma 6.8 and the fact that φ is surjective.

(ii) We have to show that each relator of $\widehat{\mathrm{Miy}}_{\mathcal{Y}}(A, \mathcal{I}, \Omega_A)$ is killed by $\widehat{\theta}$. Consider a relator

$$r = t_{k,\chi'}^{-1} \cdot {}^{t_{i,\chi}} t_{j,\chi'} \quad \text{with} \quad (j,k) \in R_{\tau_{i,\chi}}.$$

Then by definition, we have $\tau_{k,\chi'} = {}^{\tau_{i,\chi}}\tau_{j,\chi'}$ in $\mathrm{Miy}_{\mathcal{Y}}(A, \mathcal{I}, \Omega_A)$. By Lemma 6.8, this implies that $\tau_{\psi(k),\chi'} \circ \varphi = {}^{\tau_{\psi(i),\chi}}\tau_{\psi(j),\chi'} \circ \varphi$. Since φ is surjective, it follows that $\tau_{\psi(k),\chi'} = {}^{\tau_{\psi(i),\chi}}\tau_{\psi(j),\chi'}$ in $\mathrm{Miy}_{\mathcal{Y}}(B, \mathcal{J}, \Omega_B)$. Because this holds for all $\chi' \in \mathcal{Y}$, we have

$$(\psi(j), \psi(k)) \in R_{\tau_{\psi(i),\chi}}.$$

Now $\widehat{\theta}$ maps the given relator r to $t_{\psi(k),\chi'}^{-1} \cdot {}^{t_{\psi(i),\chi}} t_{\psi(j),\chi'}$, and by the definition of $\widehat{\mathrm{Miy}}_{\mathcal{Y}}(B, \mathcal{J}, \Omega_B)$, this element is trivial. □

The requirement that φ is surjective cannot be dropped in general, as the following generic type of example illustrates.

Example 6.10. Let $\Gamma = \{1, \sigma\}$ be the group of order 2 and let $\mathcal{Y} = \{1, \chi\}$ as in Example 6.3 above. Since there is only one non-trivial character in $\mathcal{Y}$, we will omit it from our notation and, for example, write τ_i in place of $\tau_{i,\chi}$. Let (A, I, Ω) be a Γ-decomposition algebra. The only (very weak) assumption we make, is the existence of three different $j, k, \ell \in I$ such that there is a relation ${}^{\tau_k}\tau_j = \tau_\ell$.

We will now construct another Γ-decomposition algebra (B, J, Ω') and a morphism $(\varphi, \psi)\colon (A, I, \Omega) \to (B, J, \Omega')$ such that the map $t_i \mapsto t_{\psi(i)}$ does *not* induce a group morphism between the corresponding universal Miyamoto groups.

Let $B = A \oplus M$, where M is a free R-module of rank 2 with basis $\{e, f\}$, and extend the multiplication of A to B trivially ($AM = MA = 0$). Let $\varphi\colon A \to B$ be the natural inclusion. Let $J = I \times \{1, 2\}$; we will construct two decompositions of B for each decomposition of A in Ω. Define

$$\Omega'[i,1] := (A_1^i \oplus Re, A_\sigma^i \oplus Rf) \quad \text{and}$$
$$\Omega'[i,2] := (A_1^i \oplus Rf, A_\sigma^i \oplus Re).$$

If we arbitrarily choose $c_i \in \{1, 2\}$ for each $i \in I$, then the map $\psi\colon I \to J\colon i \mapsto (i, c_i)$ will give rise to a morphism (φ, ψ) of Γ-decomposition algebras. In particular, this holds if we choose $c_j = c_k = 1$ and $c_\ell = 2$. Now consider the corresponding Miyamoto involutions $\tau_{(j,1)}$, $\tau_{(k,1)}$ and $\tau_{(\ell,2)}$ of B; then $\tau_{(j,1)}$ and $\tau_{(k,1)}$ fix the element e whereas $\tau_{(\ell,2)}$ maps e to $-e$. In particular,

$${}^{\tau_{\psi(k)}}\tau_{\psi(j)} = {}^{\tau_{(k,1)}}\tau_{(j,1)} \neq \tau_{(\ell,2)} = \tau_{\psi(\ell)}.$$

Hence the map $t_i \mapsto t_{\psi_i}$ does not induce a group morphism $\widehat{\mathrm{Miy}}_\chi(A,\mathcal{I},\Omega) \to \widehat{\mathrm{Miy}}_\chi(B,\mathcal{J},\Omega')$.

This behavior is caused by the fact that we can distort the map ψ. If we now restrict to *axial* decomposition algebras (see section 5) that are sufficiently nice with respect to the Miyamoto maps, then this type of distortion cannot occur, and $\widehat{\mathrm{Miy}}_\mathcal{Y}$ becomes a functor.

Definition 6.11. Let $(\Gamma,*)$ be a group fusion law, let $\mathcal{Y} \le \mathcal{X}_R(\Gamma)$ be a subgroup of the R-character group and let $\Phi = (X,*)$ be a fusion law with a Γ-grading. Let $\lambda\colon X \to R$ be an evaluation map and let $(A,\mathcal{I},\Omega,\alpha) \in (\Phi,\lambda)$-$\mathbf{AxDec}_R$ be an axial decomposition algebra, with axes $a_i := \alpha(i)$ for each $i \in \mathcal{I}$. By Proposition 4.6, we can also view this as a Γ-decomposition algebra (but usually *not* as an axial Γ-decomposition algebra!). For each $i \in \mathcal{I}$ and each $\chi \in \mathcal{Y}$, let $\tau_{i,\chi}$ be the corresponding Miyamoto map.

(i) We call $(A,\mathcal{I},\Omega,\alpha)$ *Miyamoto-stable* with respect to $\mathcal{Y}$ if for each $i \in \mathcal{I}$ and each $\chi \in \mathcal{Y}$, there is a permutation $\pi_{i,\chi}$ of $\mathcal{I}$ such that the pair $(\tau_{i,\chi},\pi_{i,\chi})$ is an automorphism of $(A,\mathcal{I},\Omega,\alpha)$ in (Φ,λ)-$\mathbf{AxDec}_R$. In other words, for each $i,j \in \mathcal{I}$:
   - the Miyamoto map $\tau_{i,\chi}$ permutes the axes; explicitly, $\tau_{i,\chi}(a_j) = a_{\pi_{i,\chi}(j)}$;
   - $\tau_{i,\chi}(A^j_x) = A^{\pi_{i,\chi}(j)}_x$ for each $x \in X$.

   In particular, if $(A,\mathcal{I},\Omega,\alpha)$ is Miyamoto-stable, then the Γ-decomposition algebra $(A,\mathcal{I},\Omega)$ is Miyamoto-closed (see Definition 6.2(iii)).

(ii) We call $(A,\mathcal{I},\Omega,\alpha)$ *of unique type* with respect to $\mathcal{Y}$ if both the map $\alpha\colon\mathcal{I} \to A$ and the map $\mathcal{I} \to \mathrm{Hom}(\mathcal{Y},\mathrm{Aut}(A))\colon i \mapsto (\chi \mapsto \tau_{i,\chi})$ are injective. In other words, for each $i \ne j$:
   - $a_i \ne a_j$;
   - there is at least one $\chi \in \mathcal{Y}$ such that $\tau_{i,\chi} \ne \tau_{j,\chi}$.

   In particular, the assumption that α is injective implies that the permutations $\pi_{i,\chi}$ are now uniquely determined by $\tau_{i,\chi}$.

Theorem 6.12. *Let $(\Gamma,*)$ be a group fusion law and let $\chi\colon\Gamma \to R^\times$ be a group homomorphism. Let $\Phi = (X,*)$ be a fusion law with a Γ-grading and let $\lambda\colon X \to R$ be an evaluation map.*

*Let $\mathcal{C}$ be the full subcategory of (Φ,λ)-$\mathbf{AxDec}_R$ consisting of axial decomposition algebras that are Miyamoto-stable and of unique type with respect to $\mathcal{Y}$. Then $\widehat{\mathrm{Miy}}_\mathcal{Y}\colon\mathcal{C} \to$ **Grp** is a functor.*

Proof. Let $(A,\mathcal{I},\Omega,\alpha) \xrightarrow{(\varphi,\psi)} (B,\mathcal{J},\Omega',\beta)$ in $\mathcal{C}$. Notice that (φ,ψ) is also a morphism in Γ-$\mathbf{Dec}_R$. By Lemma 6.8, $\varphi \circ \tau_{i,\chi} = \tau_{\psi(i),\chi} \circ \varphi$ for all $i \in \mathcal{I}$ and all $\chi \in \mathcal{Y}$. For each $i \in \mathcal{I}$ and each $j \in \mathcal{J}$, we write $a_i := \alpha(i)$ and $b_j := \beta(j)$. Then for all $i,j \in \mathcal{I}$ and all $\chi \in \mathcal{Y}$, we have

$$\begin{aligned} b_{\psi(\pi_{i,\chi}(j))} = \varphi(a_{\pi_{i,\chi}(j)}) = \varphi(\tau_{i,\chi}(a_j)) = \tau_{\psi(i),\chi}(\varphi(a_j)) \\ = \tau_{\psi(i),\chi}(b_{\psi(j)}) = b_{\pi_{\psi(i),\chi}(\psi(j))}, \end{aligned}$$

and because β is assumed to be injective, we get

$$\psi(\pi_{i,\chi}(j)) = \pi_{\psi(i),\chi}(\psi(j)). \tag{4}$$

We will show that the map $t_{i,\chi} \mapsto t_{\psi(i),\chi}$ induces a group morphism $\widehat{\mathrm{Miy}}_\mathcal{Y}(A,\mathcal{I},\Omega) \to \widehat{\mathrm{Miy}}_\mathcal{Y}(B,\mathcal{J},\Omega')$ by showing that if $(j,k) \in R_{\tau_{i,\chi}}$, then also $(\psi(j),\psi(k)) \in R_{\tau_{\psi(i),\chi}}$. So let $(j,k) \in R_{\tau_{i,\chi}}$; then by Lemma 6.6(i),

$$\tau_{k,\chi'} = {}^{\tau_{i,\chi}}\tau_{j,\chi'} = \tau_{\pi_{i,\chi}(j),\chi'}$$

for all $\chi' \in \mathcal{Y}$. Because $(A,\mathcal{I},\Omega,\alpha)$ is of unique type with respect to $\mathcal{Y}$, this can only happen if $k = \pi_{i,\chi}(j)$. Hence, by (4) and by Lemma 6.6(i) again, also

$$\tau_{\psi(k),\chi'} = \tau_{\psi(\pi_{i,\chi}(j)),\chi'} = \tau_{\pi_{\psi(i),\chi}(\psi(j)),\chi'} = {}^{\tau_{\psi(i),\chi}}\tau_{\psi(j),\chi'}$$

for all $\chi' \in \mathcal{Y}$. We conclude that indeed $(\psi(j),\psi(k)) \in R_{\tau_{\psi(i),\chi}}$. □

Example 6.13. The previous theorem is false for the ordinary Miyamoto group $\mathrm{Miy}_{\mathcal{Y}}$, as we now illustrate. Let $n \geq 3$ be odd and consider the matrix algebra $M_n(k)$ of all $n \times n$-matrices over a field k with $\mathrm{char}(k) \neq 2$. Let $J_n := M_n(k)^+$ be the corresponding Jordan algebra; this is the commutative non-associative algebra with multiplication $A \bullet B := \frac{1}{2}(AB + BA)$.

Let E_n be the set of all primitive idempotents of J_n. These are the matrices that are diagonalizable with eigenvalues 1 with multiplicity 1 and 0 with multiplicity $n-1$. It is well known that each idempotent e in a Jordan algebra J gives rise to a decomposition of J into *Peirce subspaces*, the eigenspaces of ad_e with eigenvalues 0, $\frac{1}{2}$ and 1, and moreover, this decomposition satisfies the Jordan fusion law from Example 2.4 (for example see [Jac68]*p. 119, Lemma 1). In the case of J_n and $e \in E_n$, these eigenspaces have dimension $(n-1)^2$, $2(n-1)$ and 1, respectively. This gives J_n the structure of a primitive axial decomposition algebra $(J_n, E_n, \Omega, \mathrm{id})$ admitting a $\mathbb{Z}/2\mathbb{Z}$-grading; it is clearly of unique type.

[Jac68] Jacobson, *Structure and representations of Jordan algebras*, American Mathematical Society Colloquium Publications, Vol. XXXIX, American Mathematical Society, Providence, R.I., 1968

For each $e \in E_n$, the corresponding Miyamoto map τ_e is precisely the conjugation action of $2e-1$ on J_n; since n is odd, $2e-1 \in \mathrm{SL}_n(k)$. Hence the Miyamoto group $G = \mathrm{Miy}(J_n, E_n, \Omega)$ is isomorphic to the group generated by the elements $[2e-1] \in \mathrm{PSL}_n(k) \leq \mathrm{Aut}(J_n)$ for $e \in E_n$. Since G is a non-trivial normal subgroup of $\mathrm{PSL}_n(k)$, it is isomorphic to $\mathrm{PSL}_n(k)$ itself.

Now consider the algebra morphism

$$\varphi \colon J_n \to J_{n+2} \colon A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$$

and the map $\psi \colon E_n \to E_{n+2}$ given by restriction of φ to E_n. Then the pair (φ, ψ) is a morphism of axial decomposition algebras. However, the map $\tau_e \mapsto \tau_{\psi(e)}$ does not extend to a group homomorphism from $\mathrm{PSL}_n(k)$ to $\mathrm{PSL}_{n+2}(k)$: the product of the Miyamoto maps corresponding to the primitive idempotents $E_{11}, \dots, E_{nn}$ (where E_{ij} is the matrix that is zero everywhere except at position (i,j) where it has entry 1) is trivial in $\mathrm{PSL}_n(k)$, but the product of their images under ψ is equal to the element $[\mathrm{diag}(1,1,\dots,1,-1,-1)] \in \mathrm{PSL}_{n+2}(k)$.

Notice that, in contrast, the *universal* Miyamoto group always has a quotient isomorphic to $\mathrm{SL}_n(k)$. (Determining the precise structure of the universal Miyamoto group seems to be a challenging problem.)

7. Decomposition algebras from representations

In this section we will see how representation theory directly gives rise to interesting decomposition algebras. We will assume that our base ring is the field $\mathbb{C}$ of complex numbers.

So let A be any finite-dimensional $\mathbb{C}$-algebra. Let H be any finite subgroup of the automorphism group of A and let $\mathrm{Irr}(H)$ be its representation fusion law as in Example 2.13. If A is semisimple as a $\mathbb{C}H$-module, then its unique decomposition into H-isotypic components will be an $\mathrm{Irr}(H)$-decomposition of A.

Definition 7.1 (See [Ser77])**.** Let H be a finite group and let A be a semisimple $\mathbb{C}H$-module. Let $V_1 \oplus \cdots \oplus V_n$ be a decomposition of A into irreducible modules. Denote the irreducible character of H corresponding to V_i by χ_i. For each $\chi \in \mathrm{Irr}(H)$, the submodule

[Ser77] Serre, *Linear representations of finite groups*, Springer-Verlag, New York-Heidelberg, 1977, Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42

$$A_\chi := \bigoplus_{\chi_i = \chi} V_i$$

is called the *isotypic component* of A corresponding to χ. The decomposition

$$A = \bigoplus_{\chi \in \mathrm{Irr}(H)} A_\chi$$

is called the *H-isotypic decomposition* of A; it is uniquely determined by A and H. The module A is called *multiplicity-free* if each isotypic component is irreducible; that is if $\chi_i \neq \chi_j$ for all $i \neq j$.

Theorem 7.2. *Let A be a $\mathbb{C}$-algebra. Let H be any finite subgroup of the automorphism group of A and let $(\mathrm{Irr}(H), *)$ be its representation fusion law. Let $\{H_i \mid i \in \mathcal{I}\}$ be a set of (some or all) conjugates of H in $\mathrm{Aut}(A)$ indexed by some set $\mathcal{I}$. Then:*

(i) *The H-isotypic decomposition $A = \bigoplus_{\chi \in \mathrm{Irr}(H)} A_\chi$ of A is an $(Irr(H), *)$-decomposition.*
(ii) *If A is multiplicity-free (as a $\mathbb{C}H$-module), then any non-zero element $a \in A_1$ is an axis for this decomposition.*
(iii) *For each $i \in \mathcal{I}$, let $A = \bigoplus_{\chi \in \mathrm{Irr}(H)} A^i_\chi$ be the H_i-isotypic decomposition of A. Let $\Omega = \big((A^i_\chi)_{\chi \in \mathrm{Irr}(H)} \mid i \in \mathcal{I}\big)$. Then $(A, \mathcal{I}, \Omega)$ is an $(\mathrm{Irr}(H), *)$-decomposition algebra.*
(iv) *If A is multiplicity-free (as a $\mathbb{C}H$-module) and for each $i \in \mathcal{I}$, a_i is a non-zero element of A^i_1. Then $(A, \mathcal{I}, \Omega, \alpha)$ is an axial decomposition algebra, where $\alpha\colon \mathcal{I} \to A\colon i \mapsto a_i$.*

Proof.

(i) Let $V_1 \oplus \cdots \oplus V_n$ be a decomposition of A into irreducibles. By Schur's lemma [Ser77]*§2.2, $\mathrm{Hom}(V_i \otimes V_j, V_k) = 0$ whenever χ_k is not a constituent of $\chi_i \otimes \chi_j$. Hence the projection of $V_i \cdot V_j$ onto V_k is zero.
(ii) Note that the requirement that A is multiplicity-free implies that each A_χ is a simple $\mathbb{C}H$-module. The fusion law implies that $A_1 A_\chi \subseteq A_\chi$ for all $\chi \in \mathrm{Irr}(H)$. Thus for any non-zero $a \in A_1$, we have $(\mathrm{ad}_a)_{A_\chi} \in \mathrm{Hom}(A_\chi, A_\chi)$. Schur's lemma now implies that $\mathrm{Hom}(A_\chi, A_\chi) \cong \mathbb{C}$ and hence a is an axis for this decomposition.
(iii) This follows from (i).
(iv) This follows from (ii). ☐

[Ser77] Serre, *Linear representations of finite groups*, Springer-Verlag, New York-Heidelberg, 1977, Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42

Example 7.3. A typical choice for H is the centralizer $C_G(g)$ of an automorphism $g \in G := \mathrm{Aut}(A)$ of finite order n. Notice that by Proposition 3.6, this implies that the fusion law $(\mathrm{Irr}(H), *)$ is $\mathbb{Z}/n\mathbb{Z}$-graded. Moreover, by restricting the characters of H to $\langle g \rangle \leq Z(H)$, we see that the Miyamoto maps corresponding to the decomposition $A = \bigoplus_{\chi \in \mathrm{Irr}(H)} A_\chi$ with respect to this $\mathbb{Z}/n\mathbb{Z}$-grading are precisely the elements of $\langle g \rangle \leq \mathrm{Aut}(A)$.

For example, if A is the Griess algebra, we can recover its structure as an axial algebra (with $\mathbb{Z}/2\mathbb{Z}$-grading) by taking H equal to the centralizer of a $2A$-involution and the Miyamoto group of this axial algebra is precisely the group generated by all those $2A$-involutions, i.e., the Monster group.

Conversely, we can use this technique to refine the fusion law of a decomposition.

Proposition 7.4. *Let $A = \bigoplus_{x \in X} A_x$ be a decomposition of a $\mathbb{C}$-algebra A. Let $H \leq \mathrm{Aut}(A)$ be a finite subgroup such that each A_x is H-invariant. For each $x \in X$, let χ_x be the character of the $\mathbb{C}H$-module A_x. Consider the map*

$$*\colon X \times X \to 2^X\colon (x, y) \mapsto \{z \in X \mid \langle \chi_z, \chi_x \chi_y \rangle \neq 0\}$$

where $\langle\ ,\ \rangle$ is the inner product on the space of class functions of H. Then $A = \bigoplus_{x \in X} A_x$ is an $(X, *)$-decomposition of A.

Proof. This follows immediately from the fact that $\mathrm{Hom}_{\mathbb{C}H}(A_x \otimes A_y, A_z) = 0$ whenever $\langle \chi_z, \chi_x \chi_y \rangle = 0$. ☐

Remark 7.5. Although we formulated the results in this section for a finite group H, they can easily be generalized to Lie groups or linear algebraic groups. The proof only requires a suitable version of semi-simplicity and Schur's lemma.

8. Norton algebras

If G is the Miyamoto group of a Miyamoto-stable axial decomposition algebra, then G has a natural permutation action on the set of axes. We give a reverse construction. Starting from a transitive permutation representation of a group G, we construct a Miyamoto-stable axial decomposition algebra on which G acts by automorphisms. More precisely, we will prove that Norton algebras are axial decomposition algebras. Norton algebras, in the sense of this section, were first introduced in [CGS78] starting from association schemes. We refer to [BI84] for more information about association schemes and Norton algebras.

[CGS78] Cameron, Goethals, and Seidel, *The Krein condition, spherical designs, Norton algebras and permutation groups*, Nederl. Akad. Wetensch. Indag. Math. **40** (1978), no. 2, 196–206

[BI84] Bannai and Ito, *Algebraic combinatorics. I*, The Benjamin/Cummings Publishing Co., Inc., Menlo Park, CA, 1984, Association schemes

Definition 8.1. Let X be a finite set and let $R_i \subseteq X \times X$ for $i = 0, \dots, d$. Assume:

(I) $X \times X = R_0 \cup \dots \cup R_d$ and $R_i \cap R_j = \varnothing$ for all $i \neq j$; that is, the sets R_i form a partition of $X \times X$;
(II) $R_0 = \{(x,x) \mid x \in X\}$;
(III) for each i, ${}^tR_i \coloneqq \{(x,y) \mid (y,x) \in R_i\} = R_{i'}$ for some i';
(IV) for any $(x,y) \in R_k$, the number of $z \in X$ for which $(x,z) \in R_i$ and $(z,y) \in R_j$ is a constant p^k_{ij} only depending on i, j, k;
(V) $p^k_{ij} = p^k_{ji}$ for all i, j, k.

Then $(X, \{R_i\}_{0\le i\le d})$ is called a (commutative) *association scheme*. If ${}^tR_i = R_i$ for all i, then we call the association scheme *symmetric*.

Example 8.2 ([BI84]*§II.2, Example 2.1)**.** Let G be a transitive permutation group acting on a finite set Ω. Denote the orbits of G on $\Omega \times \Omega$ by $\Lambda_0, \dots, \Lambda_d$ where $\Lambda_0 = \{(x,x) \mid x \in \Omega\}$. Then $(\Omega, \{\Lambda_i\}_{0\le i\le d})$ satisfies (I)–(IV). Requirement (V) is satisfied if and only if the corresponding permutation character is multiplicity free. This association scheme is symmetric if and only if for any i and for any $x, y \in \Lambda_i$ there exists a $g \in G$ such that ${}^gx = y$ and ${}^gy = x$. If this condition is satisfied, we say that G acts generously transitively on Ω.

Definition 8.3. Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be an association scheme.

(i) For each i, let A_i be the matrix whose rows and columns are indexed by the set X and such that
$$(A_i)_{xy} = \begin{cases} 0 & \text{if } (x,y) \notin R_i, \\ 1 & \text{if } (x,y) \in R_i. \end{cases}$$
Then $A_0 = I$ and $A_iA_j = \sum_{k=0}^d p^k_{ij} A_k$ for all i, j. Hence, by (V), they span a commutative subalgebra of the full matrix algebra. This algebra is called the *Bose–Mesner algebra* or the *adjacency algebra*. This algebra is also closed under the entry-wise or *Hadamard matrix product* which we denote by $\circ$: $(A \circ B)_{ij} = (A_{ij}B_{ij})$.

(ii) Let V be the Hermitian space with orthonormal basis $\{e_x \mid x \in X\}$ indexed by the set X. Then the A_i act naturally on V and because they pairwise commute, they can be diagonalized simultaneously by a unitary matrix U. Let $V = V_0 \oplus V_1 \oplus \dots \oplus V_r$ be the decomposition of V into common eigenspaces. It is readily verified that we can pick $V_0 = \langle (1, \dots, 1) \rangle$. Denote the matrix form, with respect to the basis $\{e_x \mid x \in X\}$, of the projection π_i of V onto V_i by E_i. Then $r = d$ and $E_0, \dots, E_d$ form a basis of primitive idempotents for the adjacency algebra of $\mathcal{X}$ [BI84]*§2.3, Theorem 3.1. Since the adjacency algebra is closed under the Hadamard product, there exist constants q^k_{ij} such that $E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^d q^k_{ij} E_k$. We call q^k_{ij} the *Krein parameters* of $\mathcal{X}$.

(iii) For each i, j and k we can define a bilinear map $\sigma^k_{ij} : V_i \times V_j \to V_k$ as point-wise multiplication with respect to the basis $\{e_x \mid x \in X\}$ composed with projection onto V_k. That is,

$$\sigma_{ij}^k(v,w) := \sum_{x\in X} \langle v, e_x\rangle\langle w, e_x\rangle \pi_k(e_x).$$

In particular, σ_{ii}^i gives V_i the structure of a commutative non-associative algebra, which is called a *Norton algebra*. We denote this product on V_i by $\star$.

Remark 8.4. If $\mathcal{X}$ is a symmetric association scheme then all the matrices A_i will be symmetric and hence simultaneously diagonalizable by a real orthogonal matrix. In that case the matrices E_i will be symmetric real matrices and the Norton algebras can be defined over $\mathbb{R}$.

Proposition 8.5 ([BI84]*§II.8, Proposition 8.3)**.** *We have*

(i) $\sigma_{ij}^k = 0$ *if and only if* $q_{ij}^k = 0$;

(ii) $\sigma_{ij}^k(\pi_i(e_x), \pi_j(e_x)) = \frac{1}{|X|} q_{ij}^k \pi_k(e_x)$.

[BI84] Bannai and Ito, *Algebraic combinatorics. I*, The Benjamin/Cummings Publishing Co., Inc., Menlo Park, CA, 1984, Association schemes

Proof. This is readily verified from $E_k(E_i \circ E_j) = \frac{1}{|X|} q_{ij}^k E_k$. □

Norton algebras provide a rich source of examples of decomposition algebras:

Theorem 8.6. *Let* $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ *be a symmetric association scheme. Let* e_x *and* π_i *be as in Definition 8.3. Let* V_i *be one of its Norton algebras and suppose* $\pi_i(e_x)$ *is non-zero for all* $x \in X$*. Then for each* $x \in X$,

$$\mathrm{ad}_{\pi_i(e_x)} \colon V_i \to V_i \colon v \mapsto \pi_i(e_x) \star v$$

is diagonalizable. Let $\bigoplus_{\lambda\in\Lambda}(V_i)_\lambda^x$ *be the decomposition of* V_i *into eigenspaces for* $\mathrm{ad}_{\pi_i(e_x)}$*. Let* $\Omega := \big(((V_i)_\lambda^x)_{\lambda\in\Lambda} \mid x \in X\big)$*. Then* $(V_i, X, \Omega, x \mapsto \pi_i(e_x))$ *is an axial decomposition algebra.*

Proof. Consider the linear operator

$$\theta \colon V \to V \colon v \mapsto \sum_{y\in X} \langle \pi_i(e_x), e_y\rangle\langle \pi_i(v), e_y\rangle\, \pi_i(e_y).$$

Its restriction to V_i equals $\iota \circ \mathrm{ad}_{\pi_i(e_x)}$, where $\iota \colon V_i \to V$ is the natural embedding. Since V_i is an invariant subspace of θ, it suffices to prove that θ is diagonalizable. The matrix form of θ with respect to the basis $\{e_x \mid x \in X\}$ is $E_i \operatorname{diag}(\pi_i(e_x))E_i$. Since $\mathcal{X}$ is symmetric, this is a real symmetric matrix and hence θ is a Hermitian operator on V and therefore θ is diagonalizable. The remaining statement is obvious. □

Remark 8.7. If $\mathcal{X}$ is not symmetric, then θ will not necessarily be a Hermitian operator. However, it can still be interesting to look at the decomposition of V_i into generalized eigenspaces of $\mathrm{ad}_{\pi_i(e_x)}$.

In the next example, we illustrate how to obtain a suitable fusion law using Proposition 7.4 and Theorem 7.2.

Example 8.8. Let G be a group and X a conjugacy class of elements of order n. Suppose that G acts generously transitively on X and consider the corresponding symmetric association scheme. Let V_i be one of its Norton algebras. The natural permutation action of G on X induces algebra automorphisms on this Norton algebra. Hence there exists a morphism $\rho \colon G \to \mathrm{Aut}(V_i) \le \mathrm{GL}(V_i)$. Let $C_G(x)$ be the centralizer in G of $x \in X$ and $(\mathrm{Irr}(C_G(x)), *)$ its representation fusion law. Since the action of $C_G(x)$ commutes with the linear operator $\mathrm{ad}_{\pi_i(e_x)}$, it leaves invariant its eigenspaces. Now apply Proposition 7.4 with $H = C_G(x)$ to construct a fusion law $(\Lambda, *')$ for the decomposition $\bigoplus_{\lambda\in\Lambda}(V_i)_\lambda^x$ of V_i. Now let $(V_i)_\lambda^x = \bigoplus_{j\in J}(V_i)_{\lambda,j}^x$ be the decomposition of

$(V_i)^x_\lambda$ into irreducible subrepresentation for $C_G(x)$. Denote the irreducible character of $C_G(x)$ corresponding to $(V_i)^x_{\lambda,j}$ as χ_j and define

$$(V_i)^x_{\lambda,\chi} = \bigoplus_{\chi_j=\chi}(V_i)^x_{\lambda,j}.$$

By Theorem 7.2, the decomposition

$$V_i = \bigoplus_{\substack{\lambda\in\Lambda\\ \chi\in\mathrm{Irr}(C_G(x))}}(V_i)^x_{\lambda,\chi}$$

is a $(\Lambda\times\mathrm{Irr}(C_G(x)),\bullet)$-decomposition, where $(\Lambda\times\mathrm{Irr}(C_G(x)),\bullet)$ is the product of the fusion laws $(\Lambda,*')$ and $(\mathrm{Irr}(C_G(x)),*)$. Since $\langle x\rangle \le Z(C_G(x))$ the map $(\lambda,\chi)\mapsto\chi(x)$ defines a $\mathbb{Z}/n\mathbb{Z}$ grading of this fusion law. The Miyamoto involution τ_x with respect to this $\mathbb{Z}/n\mathbb{Z}$-grading is precisely the automorphism $\rho(x)$ and the Miyamoto group is $\langle\rho(x)\mid x\in X\rangle = \rho(\langle X\rangle)$. In particular, if G is simple, then the Miyamoto group coincides with $\rho(G)\cong G$.

Appendix A. The category of decomposition algebras

We now explore some more advanced categorical properties of decomposition algebras. The reader who is less acquainted with the terminology and concepts may consult the excellent text book by Tom Leinster [Lei14].

[Lei14] Leinster, *Basic category theory*, Cambridge Studies in Advanced Mathematics, vol. 143, Cambridge University Press, Cambridge, 2014

Fix a commutative ring R and a fusion law $\Phi=(X,*)$ and let Φ-$\mathbf{Dec}_R$ be as in Definition 4.5.

Remark A.1. The category Φ-$\mathbf{Dec}_R$ has an initial object $(0,\varnothing,\varnothing)$ and a terminal object $(0,\{*\},(0))$. This category admits two obvious forgetful functors, namely

$$\Phi\text{-}\mathbf{Dec}_R\to\mathbf{Alg}_R\colon(A,\mathcal{I},\Omega)\rightsquigarrow A \qquad\text{and}$$
$$\Phi\text{-}\mathbf{Dec}_R\to\mathbf{Set}\colon(A,\mathcal{I},\Omega)\rightsquigarrow\mathcal{I}.$$

The corresponding left adjoints are given by

$$\mathbf{Alg}_R\to\Phi\text{-}\mathbf{Dec}_R\colon A\rightsquigarrow(A,\varnothing,\varnothing) \qquad\text{and}$$
$$\mathbf{Set}\to\Phi\text{-}\mathbf{Dec}_R\colon\mathcal{I}\rightsquigarrow\big(0,\mathcal{I},(0\mid i\in\mathcal{I})\big),$$

respectively.

Proposition A.2. *The category Φ-$\mathbf{Dec}_R$ is complete.*

Proof. Recall that a category is complete if it contains all (small) limits. From the existence theorem for limits it is sufficient to show that Φ-$\mathbf{Dec}_R$ has equalizers and all products; see, e.g., [Lei14, Proposition 5.1.26].

We begin by showing the existence of products. Let $(A_j,\mathcal{I}_j,\Omega_j)$ be a set of decomposition algebras indexed by some set J. The forgetful functors of Remark A.1 preserve limits and hence if the product of $(A_j,\mathcal{I}_j,\Omega_j)$ exists it must consist of the algebra $\prod_{j\in J}A_j$ and the index set $\prod_{j\in J}\mathcal{I}_j$. Let Π be a set of decompositions indexed by $\prod_{j\in J}\mathcal{I}_j$, where

$$\Pi[(i_j)_{j\in J}] = \Big(\prod_{j\in J}(A_j)^{i_j}_x\ \Big|\ x\in X\Big)$$

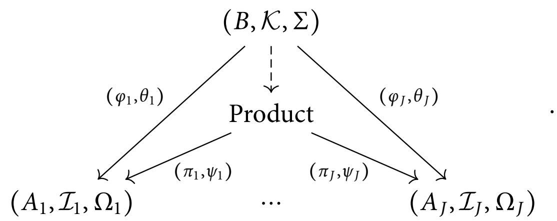
Let $\pi_k\colon\prod_{j\in J}A_j\to A_k$ and $\psi_k\colon\prod_{j\in J}\mathcal{I}_j\to\mathcal{I}_k$ be the natural projections of algebras and sets respectively: we will show that (π_k,ψ_k) is the product of the set of decomposition algebras $(A_j,\mathcal{I}_j,\Omega_j)$.

Firstly, if $\mathbf{i}=(i_j)_{j\in J}\in\prod_{j\in J}\mathcal{I}_j$ and $x\in X$ then

$$\pi_k\big(\Pi[\mathbf{i}]_x\big) = \pi_k\Big(\prod_{j\in J}(A_j)^{i_j}_x\Big) = (A_k)^{i_k}_x = (A_k)^{\psi_k(\mathbf{i})}_x$$

and so (π_k, ψ_k) is a morphism in Φ-$\mathbf{Dec}_R$.

Next we need to show that for any cone $(\varphi_j, \theta_j)\colon (B, \mathcal{K}, \Sigma) \to (A_j, \mathcal{I}_j, \Omega_j)$ there is a unique morphism from $(B, \mathcal{K}, \Sigma)$ to the product making the following diagram commute



If $b \in B_x^k$ then $\varphi_j(b) \in (A_j)_x^{\theta_j(k)}$ for all $j \in J$ and hence $\big(\varphi_j(b)\big)_{j\in J} \in \prod_{j\in J} (A_j)_x^{\theta_j(k)}$. This shows that the obvious map from $(B, \mathcal{K}, \Sigma)$ to the product is actually a morphism in Φ-$\mathbf{Dec}_R$. This map clearly makes the diagram commute and the uniqueness is a consequence of the uniqueness of π_j and ψ_j in their respective categories. This completes the proof of the existence of products.

We now show that equalizers exist in Φ-$\mathbf{Dec}_R$. Let (φ_1, ψ_1) and (φ_2, ψ_2) be two morphisms of Φ-$\mathbf{Dec}_R$:

$$(\varphi_1, \psi_1), (\varphi_2, \psi_2)\colon (A, \mathcal{I}, \Omega) \to (B, \mathcal{J}, \Theta).$$

Let $\varphi\colon E \to A$ be the equalizer of φ_1 and φ_2 in $\mathbf{Alg}_R$, let $\psi\colon \mathcal{K} \to \mathcal{I}$ be the equalizer of ψ_1 and ψ_2 in **Set** and let Σ be the tuple of decompositions given by

$$\Sigma[k] = \Big(\varphi^{-1}\Big(A_x^{\psi(k)}\Big)\,\Big|\, x \in X\Big) \text{ for } k \in \mathcal{K}.$$

To see that this is indeed a tuple of decompositions: firstly, if $e \in E_x^k \cap \sum_{y\neq x} E_y^k$ then $\varphi(e) \in A_x^{\psi(k)} \cap \sum_{y\neq x} A_y^{\psi(k)} = 0$. Now since equalizers are monic we must have $e = 0$. Secondly, if $e \in E$ and $k \in \mathcal{K}$ then $\varphi(e) = \sum_{x\in X} a_x$ for some $a_x \in A_x^{\psi(k)}$. It is sufficient to show that each a_x is in the image of φ. As $e \in E$ we know that $\varphi_1(e) = \varphi_2(e)$ and hence $\sum_{x\in X} (\varphi_1(a_x) - \varphi_2(a_x)) = 0$. However $k \in \mathcal{K}$ implies that each term $\varphi_1(a_x) - \varphi_2(a_x)$ is in a distinct component of a direct sum and hence each is zero. Now since φ_1 and φ_2 act equally on a_x for each $x \in X$, each a_x must have a preimage in E.

It is clear from the definition that (φ, ψ) is a morphism of Φ-$\mathbf{Dec}_R$ so we need only check that it is the equalizer of (φ_1, ψ_1) and (φ_2, ψ_2). Let $(\gamma, \tau)\colon (F, \mathcal{L}, \Phi) \to (A, \mathcal{I}, \Omega)$ be a morphism such that $(\varphi_1, \psi_1) \circ (\gamma, \tau) = (\varphi_2, \psi_2) \circ (\gamma, \tau)$. Define (δ, σ) by

$$\begin{aligned} \delta\colon F &\to E & \qquad \sigma\colon \mathcal{L} &\to \mathcal{K} \\ f &\mapsto \varphi^{-1}(\gamma(f)) & l &\mapsto \psi^{-1}(\tau(l)). \end{aligned}$$

Then (φ, ψ) is a morphism of decomposition algebras and $(\varphi, \psi) \circ (\delta, \sigma) = (\gamma, \tau)$. Uniqueness again follows from the uniqueness of φ and ψ in $\mathbf{Alg}_R$ and **Set** respectively. This completes the proof that equalizers exist in Φ-$\mathbf{Dec}_R$ and hence that Φ-$\mathbf{Dec}_R$ is complete. □

We now turn our attention to ideals and quotients of decomposition algebras.

Definition A.3. (i) Let $(A, \mathcal{I}, \Omega)$ be a decomposition algebra and let $I \trianglelefteq A$ be an algebra ideal. For each $i \in \mathcal{I}$ and each $x \in X$, let $I_x^i := A_x^i \cap I$ and let $\Omega \cap I := \big((I_x^i)_{x\in X} \mid i \in \mathcal{I}\big)$. We call I a *decomposition ideal* of $(A, \mathcal{I}, \Omega)$ if for each $i \in \mathcal{I}$, we have $I = \bigoplus_{x\in X} I_x^i$. Notice that this implies that $(I, \mathcal{I}, \Omega \cap I)$ is an object in Φ-$\mathbf{Dec}_R$.

(ii) If I is a decomposition ideal of $(A, \mathcal{I}, \Omega)$ and $B = A/I$, then $(B, \mathcal{I}, \Sigma)$ is again a decomposition algebra (which we then call the *quotient* decomposition algebra) obtained by setting

$$B_x^i := (A_x^i + I)/I$$

for all $i \in \mathcal{I}$ and all $x \in X$, and then letting $\Sigma = \big((B_x^i)_{x\in X} \mid i \in \mathcal{I}\big)$. Notice that the condition $I = \bigoplus_{x\in X} I_x^i$ ensures that the sum $\sum_{x\in X} B_x^i$ is a direct sum.

Proposition A.4. *Let $(\varphi, \psi)\colon (A, \mathcal{I}, \Omega_A) \to (B, \mathcal{J}, \Omega_B)$ be a morphism of decomposition algebras. Then $K = \ker\varphi$ is a decomposition ideal of $(A, \mathcal{I}, \Omega_A)$ and $(K, \mathcal{I}, \Omega_K)$ is the corresponding quotient in Φ-$\mathbf{Dec}_R$.*

Conversely, if I is a decomposition ideal of $(A, \mathcal{I}, \Omega)$ and $\pi\colon A \twoheadrightarrow A/I$ is the natural projection of algebras, then $(I, \mathcal{I}, \Omega \cap I)$ is the equalizer of the epimorphism $(\pi, \mathrm{id})\colon (A, \mathcal{I}, \Omega) \twoheadrightarrow (A/I, \mathcal{I}, \Sigma)$ and the morphism $(0, \mathrm{id})$.

Proof. We begin by showing that $K = \ker\varphi$ is a decomposition ideal. Fix some $i \in \mathcal{I}$ and let $K_x^i = K \cap A_x^i$. It is clear that $K_x^i \cap \sum_{y\neq x} K_y^i = 0$ for all $x \in X$ and that $K \supseteq \sum_{x\in X} K_x^i$, thus we need only show the opposite inclusion. For any $k \in K$ we may write $k = \sum_{x\in X} a_x^i$, where each $a_x^i \in A_x^i$. It is sufficient to show that $a_x^i \in K$, but

$$\sum_{x\in X} \varphi(a_x^i) = \varphi(k) = 0$$

where each $\varphi(a_x^i) \in B_x^{\psi(i)}$ is in a different component of a direct sum. Hence $\varphi(a_x^i) = 0$ for all x.

The second part follows directly from the first part once we note that I is the algebra kernel of π. □

Remark A.5. Recall that the categorical definition of a kernel of a morphism is the equalizer of the given morphism and a zero morphism. We would like to be able to refer to the decomposition ideal $(I, \mathcal{I}, \Omega \cap I)$ in Proposition A.4 as the kernel of the projection, however since the category Φ-$\mathbf{Dec}_R$ does not contain zero morphisms the definition of kernel does not make sense. Instead, in Proposition A.4, we use $(0, \mathrm{id})$ in place of the zero morphism and in this sense the decomposition ideals (as equalizers of these morphisms) are as close to kernels as we can realistically achieve.

References

[BI84] Eïchi Bannai and Tatsuro Ito, *Algebraic combinatorics. I*, The Benjamin/Cummings Publishing Co., Inc., Menlo Park, CA, 1984, Association schemes.

[CGS78] P. J. Cameron, J.-M. Goethals, and J. J. Seidel, *The Krein condition, spherical designs, Norton algebras and permutation groups*, Nederl. Akad. Wetensch. Indag. Math. **40** (1978), no. 2, 196–206.

[DMVC18] Tom De Medts and Michiel Van Couwenberghe, *Modules over axial algebras*, Algebr. Represent. Theory (2018), To appear. `doi:10.1007/s10468-018-9844-y`.

[FLM88] Igor Frenkel, James Lepowsky, and Arne Meurman, *Vertex operator algebras and the Monster*, Pure and Applied Mathematics, vol. 134, Academic Press, Inc., Boston, MA, 1988.

[GN08] Shlomo Gelaki and Dmitri Nikshych, *Nilpotent fusion categories*, Adv. Math. **217** (2008), no. 3, 1053–1071.

[Gri82] Robert L. Griess, Jr., *The friendly giant*, Invent. Math. **69** (1982), no. 1, 1–102.

[HRS15a] J. I. Hall, F. Rehren, and S. Shpectorov, *Primitive axial algebras of Jordan type*, J. Algebra **437** (2015), 79–115.

[HRS15b] ———, *Universal axial algebras and a theorem of Sakuma*, J. Algebra **421** (2015), 394–424.

[Isa94] I. Martin Isaacs, *Character theory of finite groups*, Dover Publications, Inc., New York, 1994.

[Jac68] Nathan Jacobson, *Structure and representations of Jordan algebras*, American Mathematical Society Colloquium Publications, Vol. XXXIX, American Mathematical Society, Providence, R.I., 1968.

[Lei14] Tom Leinster, *Basic category theory*, Cambridge Studies in Advanced Mathematics, vol. 143, Cambridge University Press, Cambridge, 2014.

[NTV14] Nikolai Nadirashvili, Vladimir Tkachev, and Serge Vlăduţ, *Nonlinear elliptic equations and nonassociative algebras*, Mathematical Surveys and Monographs, vol. 200, American Mathematical Society, Providence, RI, 2014.

[Ser77] Jean-Pierre Serre, *Linear representations of finite groups*, Springer-Verlag, New York-Heidelberg, 1977, Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

[Tka19a] Vladimir G. Tkachev, *Spectral properties of nonassociative algebras and breaking regularity for nonlinear elliptic type pdes*, Algebra I Analiz **31** (2019), no. 2, 51–74.

[Tka19b] ———, *V.M. Miklyukov: from dimension 8 to nonassociative algebras*, Math. Phys. Comp. Simul. **22** (2019), no. 2, To appear; available on arxiv:1904.08162.

Tom De Medts, Department of Mathematics: Algebra and Geometry, Ghent University, 9000 Gent, Belgium

Email address: `tom.demedts@ugent.be`

Simon F. Peacock, School of Mathematics, University of Bristol, Bristol, BS8 1TW, UK, and the Heilbronn Institute for Mathematical Research, Bristol, UK

Email address: `simon.peacock@bristol.ac.uk`

Sergey Shpectorov, School of Mathematics, Watson Building, University of Birmingham, Edgbaston, Birmingham, B15 2TT, UK

Email address: `s.shpectorov@bham.ac.uk`

Michiel Van Couwenberghe, Department of Mathematics: Algebra and Geometry, Ghent University, 9000 Gent, Belgium, Ph. D. fellowship of the Research Foundation - Flanders (FWO)

Email address: `michiel.vancouwenberghe@ugent.be`