On separable equivalence of finite dimensional algebras

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School of Mathematics

Abstract

Separable equivalence of algebras was introduced by Markus Linckelmann in [Lin11b] and may be considered as an extension to the more well-known concepts of Morita, stable and derived equivalence. We will generalise the idea of separable equivalence of algebras to additive categories and demonstrate how a separable equivalence between algebras provides separable equivalences between several related categories.

We will prove that there are several properties of an algebra that are invariant under separable equivalence. Specifically we show that if two algebras are separably equivalent then they must have the same complexity. We also show that the representation type of an algebra is preserved, including the finer grain classes of domestic and polynomial growth.

Finally, if G is a finite group with elementary abelian Sylow p-subgroup P, then we use the separable equivalence of kG and kP to provide an upper bound for the representation dimension of kG, where k is an algebraically closed field of characteristic p.

Acknowledgements

I would like to first thank my supervisor Jeremy Rickard. He has been a great support academically and a very entertaining pub companion. I cannot imagine how I would have made it through the last 4 years without him. I would however like to take this opportunity to remind him that he still owes me a yard of ale.

Thanks to EPSRC for providing the funding that made this research possible.

I have had many interesting and useful discussions with people in the School of Mathematics but I would like to give special thanks to Andrew Poulton and Martin Hauge for their extensive and continuing support. Thanks also to those who have shared an office with me over the years as they made the environment fun even when the work was at its hardest. I should also apologise to those same people for my disruptive influence; the work will no doubt be much easier in my absence.

My time in Bristol has been fantastic and this is mostly due to all the friends I have made along the way. Thanks to the Latin and Ballroom dance society for being a second family to me; and to Catherine and Charlie for putting up with such a difficult dance partner.

Finally and most importantly, thanks to my parents for making me and my four wonderful brothers: I love you all.

Author's Declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

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Contents

1 Introduction

In the area of representation theory we hope to gain an understanding of an algebra through its possible actions on abelian groups. Such an abelian group (upon which the algebra acts) is called a module for the algebra and so we may equally say that representation theory is the study of the algebra's module category. It follows that two algebras are the same, from the point of view of representation theory, if their module categories are equivalent. When this is the case the algebras are called Morita equivalent, after Kiiti Morita who did much of the early work in this area (see [Mor58]). In chapter 2 we recall some preliminary results in the representation theory of algebras.

[Mor58] Morita, *Duality for modules and its applications to the theory of rings with minimum condition*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 83–142

Starting from the module category we may define other categories of importance: the stable category and the derived category. Through equivalences of these categories we may define the concepts of stably equivalent and derived equivalent algebras. We may think of stable equivalence as a generalisation of derived equivalence and similarly, derived equivalence as a generalisation of Morita equivalence. These ideas are covered in greater detail in chapter 3.

The notion of separable equivalence was introduced by Markus Linckelmann in [Lin11b] and in many ways we may consider this idea as a generalisation of the previously mentioned equivalences. The main difference between separable equivalence and its predecessors is that separable equivalence is not defined in terms of an equivalence of categories but rather from a pair of bimodules. We first introduce Linckelmann's definition of separable equivalence in chapter 4 and then give a new generalisation of the idea for additive categories (see definition 4.16). We go on to show how a separable equivalence between two algebras provides us with further separable equivalences between related categories and use these ideas to demonstrate new ways in which we may show algebras are not separably equivalent.

[Lin11b] Linckelmann, *Finite generation of Hochschild cohomology of Hecke algebras of finite classical type in characteristic zero*, Bull. Lond. Math. Soc. **43** (2011), no. 5, 871–885

The subject of chapter 5 is the representation type of an algebra, which is a measure of the complexity of the algebra's module category. An algebra of finite-type has only finitely many isomorphism classes of indecomposable modules. To properly describe an algebra of wild-type we would need families of modules with arbitrarily large numbers of parameters. In between these two types we have algebras of tame representation type, where only 1-parameter families are required. Ju. A. Drozd proved the remarkable theorem in [Dro80] (or [Dro77] if you prefer Russian) that these three are the only possible types. Linckelmann showed in [Lin11b] that if symmetric algebras are separably equivalent then they have the same representation type. In chapter 5 we generalise this result to arbitrary algebras and go on to show that separable equivalence also preserves the subdivisions of tame: domestic and polynomial growth (see theorems 5.7 and 5.18).

[Dro80] Drozd, *Tame and wild matrix problems*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math., vol. 832, Springer, Berlin, 1980, pp. 242–258

[Dro77] Drozd, *Tame and wild matrix problems*, Matrix problems (Russian), Akad. Nauk Ukrain. SSR Inst. Mat., Kiev, 1977, pp. 104–114

[Lin11b] Linckelmann, *Finite generation of Hochschild cohomology of Hecke algebras of finite classical type in characteristic zero*, Bull. Lond. Math. Soc. **43** (2011), no. 5, 871–885

Theorem:

If A and B are separably equivalent algebras over an algebraically closed field k then:

(a) A is of finite type if and only if B is of finite type;

(b) A is domestic if and only if B is domestic;

(c) A is of polynomial growth if and only if B is of polynomial growth;

(d) A is of tame representation type if and only if B is of tame representation type;

(e) A is of wild representation type if and only if B is of wild representation type.

Related to the representation type of an algebra is its representation dimension. Maurice Auslander introduced the idea of representation dimension in [Aus71] to measure how far an algebra is from being of finite representation type. In the final chapter we give the definition of representation dimension. Focusing on elementary abelian groups P, we define a generator M of kP and calculate an upper bound for the global dimension of $\mathrm{End}(M)$. We then show that if G is any finite group with a Sylow p-subgroup isomorphic to P then the separable equivalence between kP and kG means that the global dimension of the endomorphism ring of M is an upper bound for the representation dimension of kG (see theorem 6.18).

[Aus71] Auslander, *Representation dimension of Artin algebras*, Queen Mary College Mathematics Notes (1971), 70,

Theorem:

If G is a group with elementary abelian Sylow p-subgroup P, and k is an algebraically closed field of characteristic p then

$$\operatorname{rep\,dim} kG \leq |P|.$$

Using the same ideas we are also able to calculate explicit bounds for the representation dimension of group algebras of groups with small order Sylow 2- or 3-subgroups. We rely heavily on computational methods and *Magma* for these calculations.

2 Basic concepts

In this chapter we present some fundamental concepts of the representation theory of finite dimensional algebras. We will also establish some conventions and notation that will be in use throughout the remainder of the text. A thorough understanding of this material will be essential for the reader who wishes to gain insight from any of the remaining chapters.

Conventions

- All rings considered are associative and unital unless specifically stated otherwise.
- Similarly, unless we specify otherwise, all groups will be finite and written multiplicatively.
- All R-modules will be finitely generated over R unless we specify otherwise.
- The notation ${}_RL$, M_S, ${}_RN_S$ denotes that L is a left R-module, M is a right S-module and N is a left R–right S–bimodule.
- At times we will use categorical ideas: the reader should be familiar with categories, functors, natural transformations and adjunctions.
- All the categories, functors and natural transformations we consider are additive. In fact most of the time the context will involve a commutative ring R or a field k, in which case categories will be R-additive or k-additive respectively.

2.1 Algebras

We begin with the definition of a general (associative) algebra, however most of the time we will be assuming our algebras are finite dimensional over a field.

Definition 2.1 (Algebra)**.** Let R be a commutative ring (with identity). An R*-algebra* is a ring A that is itself an R-module in such a way that the ring and module multiplications commute:

$$(ab)r = a(br) = (ar)b \qquad \text{for all } a, b \in A \text{ and } r \in R.$$

As stated above we will often be dealing with algebras over a field k, in which case we will refer to A as a k-algebra. In this instance we can replace the term R-module in the definition with k-vector space.

We present here three of the more commonly occurring examples of algebras: *Group Algebras*, *Polynomial Algebras* and *Path Algebras*. The last of these requires some further definitions and so there will be a brief digression before it is presented.

Group and Polynomial Algebras

Let G be a (finite) group and k a field. We define the *group algebra* kG, to be the free vector space over G:

$$kG = \{\sum_{g \in G} \lambda_g g \,|\, \lambda_g \in k\}.$$

The multiplication is then given by linearly extending the multiplication of the group. The unit of this algebra is $1_{kG} = 1_k 1_G$, where 1_k is the unit of the field k and 1_G is the identity in the group G; where there is no ambiguity the subscripts will be dropped.

The standard polynomial ring $k[X_1, \ldots, X_n]$ over a field k is an infinite dimensional k-algebra. We may obtain a finite dimensional factor algebra from the polynomial ring by taking the quotient by an ideal that contains a (positive) power of each indeterminate. For instance the algebra ${}^{k[X]}\!/\!_{(X^n)}$ is n-dimensional.

If k is a field of characteristic p we have an isomorphism between the p-dimensional truncated polynomial algebra and the group algebra for the cyclic group with p elements, $C_p = \langle g \,|\, g^p \rangle$.

$$\begin{array}{ccc} kC_p & \xrightarrow{\sim} & \dfrac{k[X]}{(X^p)} \\ g & \mapsto & X - 1 \end{array}$$

where (X^p) is the ideal of $k[X]$ generated by the monomial X^p.

In fact for any abelian p-group we have an isomorphism of the group algebra to a factor algebra of a polynomial ring. Specifically, if we have $G = C_{p^{r_1}} \times C_{p^{r_2}} \times \ldots \times C_{p^{r_n}}$

for some positive integers r_i and we denote by g_i a generator for the i-th cyclic group then we have an isomorphism

$$\begin{array}{rcl} kG & \xrightarrow{\sim} & \dfrac{k[X_1, X_2, \ldots, X_n]}{\left(X_1^{p^{r_1}}, X_2^{p^{r_2}}, \ldots, X_n^{p^{r_n}}\right)} \\ g_i & \mapsto & X_i - 1. \end{array}$$

2.2 Quivers and Path Algebras

Path algebras provide a particularly convenient source of examples of algebras. These are defined with reference to a type of directed graph known as a quiver. In addition to the quiver itself we may place relations on the paths; in this case case we essentially obtain a factor algebra of the full path algebra. In the next section we will see that every finite-dimensional algebra over an algebraically closed field is (Morita) equivalent to a path algebra with relations (see theorem 2.15), demonstrating the importance of this particular construction.

Throughout this section we will restrict ourselves to the case of path algebras over a field but much of the material can be generalised to include all Artin algebras; for further details see [ARS95].

[ARS95] Auslander, Reiten, and Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1995

Definition 2.2 (Quiver)**.** A *quiver* $Q = (V, E)$ is a directed graph. We define the maps $s: E \to V$ and $t: E \to V$ to be the source and target maps. If $\alpha \in E$ is an arrow from $e_1 \in V$ to $e_2 \in V$ then $s(\alpha) = e_1$ and $t(\alpha) = e_2$.

Definition 2.3 (Path)**.** A *path* in a quiver Q is either a sequence of arrows $p = \alpha_1 \ldots \alpha_n$ with $t(\alpha_i) = s(\alpha_{i+1})$ for each $1 \leq i < n$ or a vertex $e_i \in V$. The paths $e_i \in V$ are called *trivial paths*. The definitions of s and t are extended in the obvious way for paths, that is the source of the first arrow in the path or the target of the last.

Definition 2.4 (Path algebra)**.** The *path algebra* for a quiver Q over a field k, denoted kQ, is the k-vector space with basis the set of all paths. The ring multiplication is given by concatenation of paths, if this makes sense, and is zero otherwise. This is then extended linearly in the obvious way. Explicitly for basis elements $p = \alpha_1 \ldots \alpha_m$

and $q = \beta_1 \dots \beta_n$ in kQ the multiplication pq is defined as

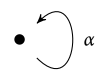
$$pq = \begin{cases} \alpha_1 \dots \alpha_m \beta_1 \dots \beta_n & \text{if } t(\alpha_m) = s(\beta_1) \\ p & \text{if } q = e_{t(p)} \\ q & \text{if } p = e_{s(q)} \\ 0 & \text{otherwise} \end{cases}$$

Note that if Q has finitely many vertices then the identity element of kQ is

$$1 = \sum_{i \in V} e_i$$

a sum of orthogonal idempotents.

Example. Consider the quiver Q,



The paths are $e, \alpha^1, \alpha^2, \dots$ and we have an isomorphism

$$\begin{array}{ccc} kQ & \xrightarrow{\sim} & k[X] \\ \alpha & \mapsto & X \end{array}$$

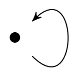
We are now in a position to consider relations amongst the paths of the quiver. In the simplest case this will amount to setting certain paths to be zero however more complicated relations are of course permitted. In order for the relations to make sense we will require that all paths involved in a given relation be between the same two vertices (and in the same direction).

Definition 2.5 (Quiver with relations)**.** A *relation* on a quiver Q is a k-linear sum of paths from a vertex a to a vertex b. That is $\sigma = \sum_n \lambda_n p_n \in kQ$ with $\lambda_n \in k$ and $a = s(p_n)$ and $b = t(p_n)$ for all n. If $\{\sigma_i \mid i \in I\}$ is a set of relations then the pair $(Q, \{\sigma_i \mid i \in I\})$ is a *quiver with relations* and its associated algebra is the quotient $kQ/\langle \sigma_i \mid i \in I \rangle$.

If we let J be the ideal generated by the arrows of Q then we will mainly consider relations for which $J^t \leq \langle \sigma_i \mid i \in I \rangle \leq J^2$ for some positive integer t. When this is the case we say $\{\sigma_i \mid i \in I\}$ is a set of *admissible relations*. The requirement that J^t

is contained in the ideal generated by the relations means that if Q is finite then we obtain a finite dimensional quotient algebra. On the other side, if we did not have the ideal contained in J^2, then we could have obtained the same algebra from a simpler quiver.

Going back to the example we had above: if we take the same quiver Q



and impose the relation α^n, then we obtain an algebra isomorphic to the truncated polynomial ring ${}^{k[X]}\!/\!{}_{(X^n)}$. As we saw earlier, if our field has characteristic p and n is equal to a power of p, then we have an isomorphism from the path algebra of the quiver with relations to that of the group algebra for a cyclic group of order n.

We note that a right module for the path algebra can be equivalently interpreted as a set of vector spaces V_i, one for each vertex of the quiver, and a set of linear maps f_α, one for each arrow, where $f_\alpha\colon V_{s(\alpha)} \to V_{t(\alpha)}$. If we have relations on the quiver then the linear maps are required to satisfy these relations in the natural way. When thought of in this way a module homomorphism, say from (V, f) to (W, g), is a collection of linear maps $h_i\colon V_i \to W_i$ such that for each arrow α, the square

$$\begin{array}{ccc} V_i & \xrightarrow{h_i} & W_i \\ {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle g_\alpha} \\ V_j & \xrightarrow{h_j} & W_j \end{array}$$

commutes.

In order to see that this is an alternative description to what we already know of as a module for the algebra we will describe the operations required to move between the interpretations.

If we have the representation given by V_i and f_α then let $V = \bigoplus_i V_i$ be the vector space direct sum of the V_i on which we put the kQ action

$$\begin{array}{rcl} kQ & \longrightarrow & \mathrm{End}_k(V) \\ e_i & \mapsto & \iota_i \pi_i \\ \alpha_1\alpha_2\ldots\alpha_n & \mapsto & \iota_{t(\alpha_n)} f_{\alpha_n} f_{\alpha_{n-1}} \cdots f_{\alpha_1} \pi_{s(\alpha_1)} \end{array}$$

where ι_i is the inclusion $V_i \hookrightarrow V$ and π_i is the projection $V \twoheadrightarrow V_i$.

If on the other hand we have a right kQ-module V, then we define the vector spaces $V_i = Ve_i$. We know that each $\alpha = e_i \alpha e_j$ for some i and j, and therefore multiplication by α induces a linear map $f_\alpha \colon V_i \to V_j$.

2.3 Some representation theory

In the study of the representation theory of an algebra A, we aim to understand the module category of A. The full module category is denoted $\operatorname{Mod} A$, whilst the full subcategory of finitely generated modules is denoted $\operatorname{mod} A$. In particular, two algebras can be considered equivalent, from the point of view of representation theory, if their module categories are equivalent. This is what is referred to as Morita equivalence. This idea gives a less granular, and often more useful, relation than is obtained from isomorphism of the algebras themselves.

In this section we recall some fundamental results in the representation theory of finite dimensional algebras, which for the most part will be included without proof. For the interested reader further details can be found in [AF92] or [ARS95], or many other sources. Note that throughout this entire section we will be assuming that algebras are finite dimensional over a field.

[AF92] Anderson and Fuller, *Rings and categories of modules*, second ed., Graduate Texts in Mathematics, vol. 13, Springer-Verlag, New York, 1992

[ARS95] Auslander, Reiten, and Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1995

Definition 2.6 (Radical)**.** Let M be a right A-module. The *radical* of M is given by the following equivalent definitions

$$\begin{aligned} \operatorname{rad} M &= \bigcap \{\ker \phi \mid \phi \colon M \to S,\, S \text{ simple}\} \\ &= \bigcap \{N \mid N \leq M,\, N \text{ maximal}\} \end{aligned}$$

Proposition 2.7 (Properties of rad)**.**

(a) $\operatorname{rad} A$ *is an ideal of A;*

(b) the quotient ${}^{A}\!/\!_{\operatorname{rad} A}$ *is semisimple;*

(c) the quotient ${}^{M}\!/\!_{\operatorname{rad} M} = \operatorname{hd} M$ *is the maximal semisimple quotient of M (called the head of M);*

(d) $\operatorname{rad} M = M \operatorname{rad} A$*;*

(e) $(\operatorname{rad} A)^n = \overbrace{\operatorname{rad}(\operatorname{rad}(\cdots \operatorname{rad} A)\cdots)}^{n\text{-times}} = 0$ *for some positive integer n;*

(f) $\operatorname{rad} A$ *is the unique maximal nilpotent ideal of A;*

(g) *if M is a nilpotent ideal of A and A/M is semisimple then* $M = \operatorname{rad} A$.

If Q is a finite quiver and $J < kQ$ is the ideal generated by all the arrows of Q, then it is clear to see that J is nilpotent. Further, we have that $kQ/J \cong \bigoplus_i e_i k$, a semisimple algebra, and so by property (g) we see that J is the radical of kQ.

Dual to the concept of the radical is the *socle* of a module.

Definition 2.8 (Socle)**.** The *socle*, $\operatorname{soc} M$, of M is given by the following equivalent definition

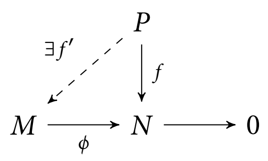
$$\begin{aligned}\operatorname{soc} M &= \sum \{\operatorname{im}\phi \mid \phi\colon S \to M, S \text{ simple}\}\\ &= \sum \{N \mid N \leq M, N \text{ simple}\}\end{aligned}$$

Projective modules

The projective modules for an algebra generalise the idea of free modules. In fact, since freeness of a module is not a categorical idea, if we only care about the representation theory of an algebra up to Morita equivalence, then projective modules are much more natural to work with.

Definition 2.9 (Projective module)**.** A (right) A-module P is called *projective* if equivalently:

- the functor $\operatorname{Hom}_A(P, —)$ is exact;
- for any surjective module map $\phi\colon M \to N$ and any module map $f\colon P \to N$ there exists $f'\colon P \to M$ such that $f = \phi f'$;



- any exact sequence $M \to P \to 0$ splits;
- P is a module direct summand of A^n for some n.

To each module M, we can assign a projective module P_M, its *projective cover*. This is the smallest projective module with M as a factor module.

Definition 2.10 (Projective cover)**.** The *projective cover* of an A-module M, is a projective A-module $P = P_M$ of minimal dimension together with a surjection $P \to M \to 0$.

Note we may refer to *the* projective cover as a consequence of the following theorem.

Theorem 2.11: *Uniqueness of projective covers*

> If $P \overset{\pi_P}{\to} M \to 0$ is a projective cover and $Q \overset{\pi_Q}{\to} M \to 0$ is any projective module mapping onto M then $Q \cong P \oplus P'$ for some $P' \leq \ker \pi_Q$. In particular projective covers are unique up to isomorphism.

This theorem is a consequence of Schanuel's lemma, see for example section 5 of [Lam99].

[Lam99] Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999

Proposition 2.12 (Properties of projective covers)**.**

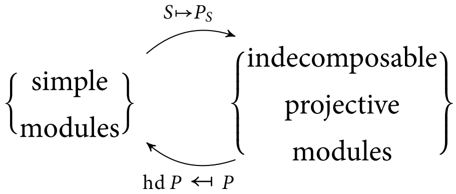
(a) If S is a simple A-module then its projective cover is indecomposable.

(b) A module and its head share a projective cover: $P_M = P_{\operatorname{hd} M}$.

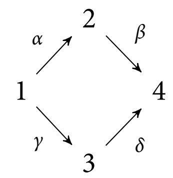
(c) The projective cover of a direct sum is the direct sum of projective covers: $P_{M \oplus N} = P_M \oplus P_N$.

(d) A simple module is isomorphic to the head of its projective cover: $S \cong \operatorname{hd} P_S$.

The above proposition gives rise to a one-to-one correspondence between simple modules and indecomposable projectives (up to isomorphism)

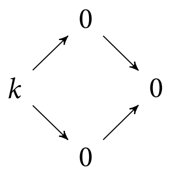


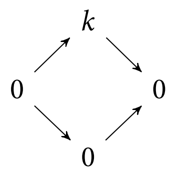
Example. Consider the algebra for the quiver Q,

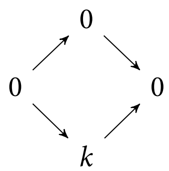


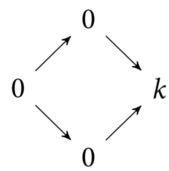
$\alpha\beta = \gamma\delta$

The simple modules are

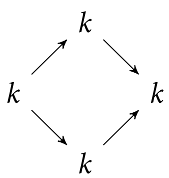
$S_1 =$ 

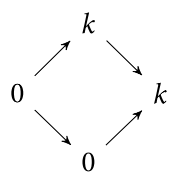
$S_2 =$ 

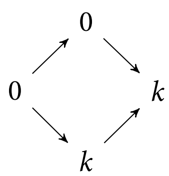
$S_3 =$ 

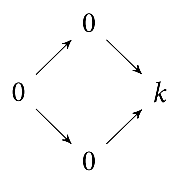
$S_4 =$ 

Using the fact that $1 = \sum_i e_i$ we can decompose the regular module $kQ = \bigoplus_i e_i kQ$ and so $P_i = e_i kQ = \langle p \mid s(p) = i \rangle$ is projective. Since ${P_i}/{\operatorname{rad} P_i} = e_i k$ is simple we see that P_i are the indecomposable projectives.

$P_1 =$ 

$P_2 =$ 

$P_3 =$ 

$P_4 =$ 

The importance of projective modules can be seen in the following theorem, which shows how to obtain the category $\operatorname{mod} A$ from the category of finite dimensional projectives.

If $\mathcal{A}$ and $\mathcal{B}$ are categories then we denote by $\operatorname{Fun}(\mathcal{A}, \mathcal{B})$ the functor category. This has as objects the functors from $\mathcal{A}$ to $\mathcal{B}$. The morphisms from a functor F to a functor G are the natural transformations, the class of which we denote by $\operatorname{Nat}(F, G)$

Theorem 2.13:

Let $\operatorname{proj} A$ denote the category of finitely generated projective A-modules.

The categories $\operatorname{mod} A$ and the functor category $\operatorname{Fun}\big((\operatorname{proj} A)^{\mathrm{op}}, \operatorname{mod} k\big)$ are equivalent under the mappings

$$\begin{array}{rcl} \operatorname{mod} A & \overset{\sim}{\longleftrightarrow} & \operatorname{Fun}\big((\operatorname{proj} A)^{\mathrm{op}}, \operatorname{mod} k\big) \\ M & \mapsto & \operatorname{Hom}_A(-, M) \\ F(A) & \leftarrow\!\shortmid & F \end{array}$$

For an A-module M, let

$$\operatorname{add} M = \{N \mid N \text{ is a summand of } M^n \text{ for some positive integer } n\}$$

so that $\operatorname{add} A = \operatorname{proj} A$.

Let $E = \operatorname{End}(M_A)$ so that

$$\operatorname{Hom}_A(M, -)\colon \operatorname{mod} A \to \operatorname{mod} E$$

is a functor from A-modules to E-modules. This functor takes $M_A \mapsto E_E$ and restricting to $\operatorname{add} M$ we have an equivalence of categories

$$\operatorname{add} M \overset{\sim}{\longrightarrow} \operatorname{proj} E.$$

Let $P_1, \ldots, P_n$ be a complete list of indecomposable projective A-modules and

$$T = P_1^{d_1} \oplus \cdots \oplus P_n^{d_n}$$

with $d_i > 0$ for all i. By the above argument we have an equivalence $\operatorname{proj} A = \operatorname{add} T \simeq \operatorname{proj} \operatorname{End} T$ and using theorem 2.13 we have

$$\operatorname{mod} A \simeq \operatorname{mod} \operatorname{End} T.$$

Conversely, given an equivalence of module categories $\operatorname{mod} A \simeq \operatorname{mod} B$, then there is some A-module X_A such that

$$X_A \longleftrightarrow B_B$$

and we must have that

$$X \cong P_1^{d_1} \oplus \cdots \oplus P_n^{d_n}$$

with each $d_i > 0$. Under this equivalence, simple modules for A map to simple modules for B and we have

$$\operatorname{Hom}_A(X, S) = \operatorname{Hom}_B(B, S') \cong S'$$

where the isomorphism is as vector spaces. If we know that every simple module S has $\operatorname{End}_A(S) \cong k$ (as would be the case if k were algebraically closed) then this demonstrates that the simple module S_i' associated with the projective P_i' in B is d_i-dimensional.

This idea of a projective module that is the direct sum of at least one copy of each indecomposable projective is as close to the concept of a free module as one can get using categorical methods. From the above discussion we see that these lead to equivalent module categories. In fact we may obtain the simplest Morita equivalent algebra to some algebra A, by taking the endomorphism ring of the direct sum of one copy of each indecomposable projective. In this situation we have an algebra where each simple module is 1-dimensional.

Definition 2.14 (Basic algebra)**.** An algebra is known as a *basic algebra* if all its simple modules are 1-dimensional.

Linking back to section 2.2 on quivers and path algebras we have:

Theorem 2.15:

Every finite dimensional basic algebra is isomorphic to a path algebra of a finite quiver with admissible relations. Moreover the quiver for such an algebra is unique, although the relations need not be.

2.4 Symmetric and separable algebras

We continue this introductory chapter with symmetric algebras and separability. These will lead to the simplest examples of separable equivalence, the main topic of this text. In order to talk about symmetric algebras we first need the idea of a dual of a module. In our context there are two different notions of a dual and these are defined in terms of Hom-sets in a similar way to that of a vector space dual.

Dual of a module

Let X_A, ${}_BY_A$ and ${}_BZ$, be respectively a right A-module, a left B–right A–bimodule and a left B-module for the R-algebras A and B. We can put a natural left B-module structure on the group $\mathrm{Hom}_A(X,Y)$ and a natural left A-module structure on the group $\mathrm{Hom}_B(Y,Z)$ as follows. In the first case if $\phi\colon X \to Y$ and $b \in B$, then we define $(b\phi)(x) = b(\phi(x))$. In the second example if we let $\theta\colon Y \to Z$ and $a \in A$, then we define $(a\theta)(y) = \theta(ya)$.

If A is an R-algebra and M a right A-module then we can use these ideas to define two different duals for M, each of which is a left A-module. Firstly

$$M^* = \mathrm{Hom}_R(M,R)$$
$$(a\phi)(m) = \phi(ma)$$

where we are considering ${}_RM_A$ as a left R–right A–bimodule, and secondly

$$M^\vee = \mathrm{Hom}_A(M,A)$$
$$(a\theta)(m) = a(\theta(m))$$

where we are considering A as the regular A-A–bimodule.

Definition 2.16 (Symmetric algebra)**.** A finite dimensional algebra A over a field k, is called *symmetric* if the following equivalent properties hold:

(a) There is a linear map $\theta\colon A \to k$ with $\theta(ab) = \theta(ba)$ and $\ker\theta$ contains no non-zero left or right ideals;

(b) $A \cong A^*$ as A-bimodules;

(c) there is an automorphism of $\mathrm{mod}\,A$ that maps M^* to $M^\vee$;

(d) for $M \in \operatorname{mod} A$, $P \in \operatorname{proj} A$ there is a vector space isomorphism $\operatorname{Hom}_A(M, P) \cong \operatorname{Hom}_A(P, M)^*$ that is functorial in both M and P.

Remarks. We make some remarks regarding this definition.

- The linear map in (a) is known as a symmetrizing form for the algebra A.
- If $\hat{s}\colon A \xrightarrow{\sim} A^*$ is the isomorphism given in (b), then $s = \hat{s}(1)$ is a symmetrizing form. Moreover, the image of $a \in A$ under the same isomorphism is the map $s_a\colon A \to k$, defined by $s_a(b) = s(ab)$; thus s determines $\hat{s}$.
- The automorphism group of A as an A-A–bimodule is canonically isomorphic to the group of units of the centre of A.

$$\begin{array}{ccc} \operatorname{Aut}(A) & \xrightarrow{\sim} & Z(A)^\times \\ \varphi & \mapsto & \varphi(1) \end{array}$$

  Any other symmetrizing form of A is given by s_z for some $z \in Z(A)^\times$.
- Property (c) shows that for symmetric algebras the two duals we defined above coincide.
- Notice that (d) is a condition purely in terms of the modules of an algebra, therefore if two algebras are Morita equivalent then one is symmetric if and only if the other is symmetric.

Example. We give two examples of symmetric algebras.

- The trace is a symmetrizing form for $\operatorname{Mat}_n(k)$, the algebra of square matrices with entries from k.
- For a finite group G, the group algebra kG is symmetric with symmetrizing form $\sum \lambda_g g \mapsto \lambda_1$.

Definition 2.17 (Separable algebra)**.** An R-algebra A is called *separable* if it satisfies the follow equivalent properties

(a) A is projective as an A-A–bimodule;

(b) The multiplication morphism

$$\begin{array}{ccc} \mu\colon A \otimes_R A & \longrightarrow & A \\ a \otimes a' & \mapsto & aa' \end{array}$$

splits as a morphism of A-A–bimodules.

(c) There is an element

$$e = \sum_i e_i \otimes e_i' \in A \otimes A$$

such that

$$\mu(e) = \sum_i e_i e_i' = 1$$

and $ae = ea$ for all $a \in A$: that is $\sum ae_i \otimes e_i' = \sum e_i \otimes e_i' a$.

Notice that if A is a separable algebra and M is an A-module then the functor $\mathrm{Hom}_A(M, —)$ is a summand of the functor

$$\begin{aligned}\mathrm{Hom}_A(M \underset{A}{\otimes} A \underset{k}{\otimes} A, —) &\cong \mathrm{Hom}_k(M \underset{A}{\otimes} A, \mathrm{Hom}_A(A, —)) \\ &\cong \mathrm{Hom}_k(M, —)\end{aligned}$$

which is exact and hence A is semisimple.

Example. The canonical example of a separable algebra is the matrix algebra, $\mathrm{Mat}_n(R)$; we show this using property (c) of the definition.

Let e_{ij} be the matrix with one in the (i,j)-th position and zeros elsewhere, and let $e = \sum_{i=1}^n e_{i1} \otimes e_{1i}$, so that $\sum_i e_{i1}e_{1i} = \sum_i e_{ii} = 1$.

Now

$$e_{kl}e = \sum_i e_{kl}e_{i1} \otimes e_{1i} = e_{k1} \otimes e_{1l} = \sum_i e_{i1} \otimes e_{1i}e_{kl} = ee_{kl}$$

and since $\mathrm{Mat}_n(R)$ is generated by the e_{kl} we have that $ae = ea$ for any $a \in \mathrm{Mat}_n(R)$.

2.5 Auslander–Reiten quiver

The Auslander–Reiten quiver for an algebra contains information regarding the indecomposable modules for the algebra along with the homomorphisms between these modules. In this way, one can think of the Auslander–Reiten quiver as representing the whole module category. In order to understand the definition of this quiver we must first understand almost split sequences, which are sometimes referred to as Auslander–Reiten sequences.

Definition 2.18 (Almost split)**.** A map of A-modules $\alpha\colon N \to M$ is called *right almost split* if the following two conditions are satisfied

- α is not a split epimorphism;

- if $\theta\colon X \to M$ is not a split epimorphism then there exists $\tilde{\theta}\colon X \to N$ such that $\alpha\tilde{\theta} = \theta$.

$$\begin{array}{ccc} & & X \\ & \swarrow^{\exists\tilde{\theta}} & \downarrow{\scriptstyle \theta} \\ N & \xrightarrow{\alpha} & M \end{array}$$

If additionally $\ker\alpha$ contains no (non-zero) summands of N then α is called *minimal right almost split*. The dual concept of (minimal) left almost split is defined analogously.

Definition 2.19 (Almost split sequence)**.** An exact sequence

$$0 \longrightarrow K \longrightarrow N \xrightarrow{\alpha} M \longrightarrow 0$$

is called an *almost split sequence* if α is minimal right almost split.

Theorem 2.20: *Uniqueness of minimal almost split*

> If $X \overset{\alpha}{\to} M$ is minimal right almost split and $Y \overset{\beta}{\to} M$ is right almost split (not necessarily minimal), then $Y \cong X \oplus X'$ for some $X' \leq \ker\beta$. In particular minimal right almost split maps are unique up to isomorphism.

Note the similarity between this and theorem 2.11 on the uniqueness of projective covers. We have a dual result for the uniqueness of left minimal almost split maps.

Proposition 2.21. *The property of being an almost split sequence is self dual and so if*

$$0 \longrightarrow K \xrightarrow{\beta} N \xrightarrow{\alpha} M \longrightarrow 0$$

is an almost split sequence then β is minimal left almost split.

Note that this proposition means that definition 2.19 could have been phrased in terms of left minimal almost split maps.

Lemma 2.22.

(a) If $\alpha\colon M \to N$ is minimal right almost split then N is indecomposable.

(b) If $\alpha\colon M \to N$ is minimal left almost split then M is indecomposable.

For the following theorem we need to define the transpose of a module M. Let

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be the start of a minimal projective resolution. We take the A-dual of this sequence and call the cokernel the transpose of M which we denote by $\operatorname{Tr} M$:

$$M^\vee \longrightarrow P_0^\vee \longrightarrow P_1^\vee \longrightarrow \operatorname{Tr} M$$

Theorem 2.23:

If k is an algebraically closed field and A is a k-algebra then for any non-projective indecomposable A-module M, there exist a module X and an almost split sequence

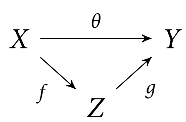
$$0 \longrightarrow \tau M \longrightarrow X \longrightarrow M \longrightarrow 0$$

where

$$\tau M = (\operatorname{Tr} M)^* = \operatorname{Hom}_k(\operatorname{Tr} M, k).$$

The existence of almost split sequences is necessary for the construction of the Auslander–Reiten quiver (and the associated Auslander algebra).

Definition 2.24 (Irreducible)**.** A map $\theta\colon X \to Y$ is called *irreducible* if it is neither a split epimorphism nor a split monomorphism and, if the diagram



commutes then either f is a split monomorphism or g is a split epimorphism.

Lemma 2.25. *Any irreducible map $\theta\colon X \to Y$ is either an epimorphism or a monomorphism.*

Proposition 2.26. *If $\alpha\colon N \to M$ is minimal right almost split then α is irreducible.*

Definition 2.27 (Radical)**.** Let $X, Y \in \operatorname{mod} A$. We define $\operatorname{rad}(X, Y) \le \operatorname{Hom}_A(X, Y)$ to be the set of maps that do not map any summand of X isomorphically to a summand of Y. That is

$$\begin{aligned}\operatorname{rad}(X,Y) = \Big\{ f\colon X \to Y \;\Big|\; & \text{if } M \text{ is indecomposable and } M \to X \text{ and } Y \to M \\ & \text{are any maps then the composition } M \to X \stackrel{f}{\to} Y \to M \text{ is} \\ & \text{not an isomorphism} \Big\}\end{aligned}$$

We define inductively $\operatorname{rad}^n(X,Y)$ as

$$\begin{aligned}\operatorname{rad}^n(X,Y) &= \Big\{ f\colon X \to Y \;\Big|\; f = gh, \text{ with } g \in \operatorname{rad}^{n-1}(Z,Y) \\ &\qquad \text{and } h \in \operatorname{rad}(X,Z) \text{ for some } Z \Big\} \\ &= \Big\{ f\colon X \to Y \;\Big|\; f = gh, \text{ with } g \in \operatorname{rad}^{n-k}(Z,Y) \\ &\qquad \text{and } h \in \operatorname{rad}^k(X,Z) \text{ for some } Z, 0 < k < n \Big\}\end{aligned}$$

If M and N are indecomposable then we have

$$\operatorname{rad}(N,M) = \begin{cases} \operatorname{Hom}_A(N,M) & M \ncong N \\ \operatorname{rad}\operatorname{End} M & M \cong N \end{cases}$$

For M, N indecomposable it is clear to see that

$$\{\alpha\colon N \to M \mid \alpha \text{ irreducible}\} = \operatorname{rad}(N,M) \smallsetminus \operatorname{rad}^2(N,M).$$

Take again the sequence $0 \to \tau M \to Y \stackrel{\alpha}{\to} M$ and consider an irreducible map $\theta\colon N \to M$ from an indecomposable module N to M.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tau N & \longrightarrow & Y & \stackrel{\alpha}{\longrightarrow} & M \\ & & & & & {\scriptstyle\beta}\nwarrow & \uparrow{\scriptstyle\theta} \\ & & & & & & N \end{array}$$

Since θ is not a split epimorphism it must factor through α and so β must be a split monomorphism. This means that N is a summand of Y and since (Krull-Schmidt) there are only finitely many summands we know that there are only finitely many N for which an irreducible map to M exists.

Now for indecomposable modules M and N, consider the space

$$\frac{\operatorname{rad}(N,M)}{\operatorname{rad}^2(N,M)}$$

with basis $\{\alpha_1, \dots, \alpha_n\}$. Since this is a basis we have $\sum_i \alpha_i \notin \text{rad}^2(N, M)$ so $\sum_i \alpha_i$ is irreducible and thus $Y = N^n \oplus Y'$. Conversely, if $Y = N^n \oplus Y'$ and $\alpha: Y \to M$ is minimal right almost split then there are $\alpha_i: N \to M$ and $\beta: Y' \to M$ such that

$$\alpha = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_n \oplus \beta.$$

Any non-zero linear combination of the α_i is irreducible and thus

$$\dim\left(\frac{\text{rad}(N, M)}{\text{rad}^2(N, M)}\right) \geq n.$$

This shows that the following definition is consistent.

Definition 2.28 (Auslander–Reiten quiver)**.** Let A be an algebra over an algebraically closed field k. The *Auslander–Reiten* quiver for A has a vertex for each isomorphism class of indecomposable A-modules. If M and N are indecomposable modules then there are exactly n edges $[N] \to [M]$ where n is equivalently given by

- $n = \dim\left(\frac{\text{rad}(N,M)}{\text{rad}^2(N,M)}\right)$;
- $N^n \oplus Y \to M$ is a minimal right almost split map and N is not a summand of Y.
- $N \to M^n \oplus X$ is a minimal left almost split map and M is not a summand of X.

The irreducible maps may satisfy relations that we can add to the Auslander–Reiten quiver and in this way we may form a new algebra from an existing one. If an algebra A has finitely many indecomposable modules, so that its Auslander–Reiten quiver is finite, then we may construct the path algebra B from the AR-quiver in the usual way. In this context B is known as the Auslander algebra of A. Modules for B are simply representations of the module category for A and we have a categorical equivalence

$$\text{Fun}(\text{mod}\, A, \text{mod}\, k) \cong \text{mod}\, B$$

Example. Consider the path algebra for

$$\bullet \longrightarrow \bullet$$

This has indecomposable modules

S_1	P_1	$S_2 = P_2$
$k \longrightarrow 0$	$k \longrightarrow k$	$0 \longrightarrow k$

We can examine the possible homomorphisms between these modules to calculate the radical spaces. For example if we have a map $\varphi\colon S_1 \to P_1$

$$\begin{array}{ccc} k & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \end{array}$$

then as the square must commute we must have that φ is the zero homomorphism. In this way we can see that the only non-zero maps that exist are $P_1 \to S_1$ and $S_2 \to P_1$:

$$\begin{array}{ccc} k & \longrightarrow & k \\ {\scriptstyle 1}\downarrow & & \downarrow \\ k & \longrightarrow & 0 \end{array} \qquad\qquad \begin{array}{ccc} 0 & \longrightarrow & k \\ \downarrow & & \downarrow{\scriptstyle 1} \\ k & \longrightarrow & k \end{array}$$

or scalar multiples of these.

The composition of these two maps is zero and so the Auslander–Reiten quiver is

$$[S_2] \xrightarrow{\alpha} [P_1] \xrightarrow{\beta} [S_1] \qquad\qquad \alpha\beta = 0.$$

3 Equivalence of algebras

We aim to introduce the concept of separable equivalence, which is an equivalence relation on algebras. This idea can be considered a generalisation of some more well-known equivalences. The mostly widely known equivalence relation on algebras is surely isomorphism, which is incredibly restrictive. In fact many consider two algebras to be literally the same if they are isomorphic—the author included.

In this chapter we will explore the idea of Morita equivalence (which has been mentioned in the preceding chapter) and introduce the idea of stable equivalence and derived equivalence. These concepts will naturally lead on to the idea of separable equivalence, which we define for the first time in the next chapter.

3.1 Morita equivalence

We begin with Morita equivalence which we have already seen in earlier sections; two algebras are Morita equivalent if they have equivalent module categories. The discussion preceding definition 2.14 demonstrates that we may have non-isomorphic algebras that are Morita equivalent. The theorem we give here provides an alternative characterisation of this equivalence, which we will then extend to the other types of equivalence.

Theorem 3.1:

The R-algebras A and B are Morita equivalent if and only if there are bimodules ${}_AP_B$ and ${}_BQ_A$ such that

(a) the modules ${}_AP$, P_B, ${}_BQ$ and Q_A are projective; and

(b) there are bimodule isomorphisms

$$ {}_AP \underset{B}{\otimes} Q_A \xrightarrow{\sim} {}_AA_A \qquad {}_BQ \underset{A}{\otimes} P_B \xrightarrow{\sim} {}_BB_B $$

[AF92] Anderson and Fuller, *Rings and categories of modules*, second ed., Graduate Texts in Mathematics, vol. 13, Springer-Verlag, New York, 1992

We will only prove that if the modules P and Q exist then the algebras are Morita equivalent. For all the gory details see theorem 22.2 of [AF92].

Proof. [⇐] We claim that the functors

$$F = - \underset{A}{\otimes} P\colon \operatorname{mod} A \longrightarrow \operatorname{mod} B \qquad G = - \underset{B}{\otimes} Q\colon \operatorname{mod} B \longrightarrow \operatorname{mod} A$$

provide inverse equivalences for the module categories. The composition GF is given by

$$GF = - \underset{A}{\otimes} P \underset{B}{\otimes} Q.$$

The isomorphism $P \underset{B}{\otimes} Q \xrightarrow{\sim} A$ then gives us

$$GF \cong - \underset{A}{\otimes} A \cong \mathrm{Id}_{\operatorname{mod} A}$$

and an identical argument works for the composition FG. □

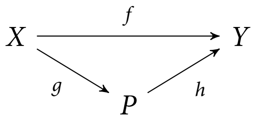
The proof of the other direction for this theorem involves taking a pair of inverse equivalences F and G, and setting $P_B = F(A_A)$. The left action of A on itself provides an action of A on $F(A_A)$ and this gives a left action of A on P. Similarly, we take $Q = G(B)$ and carry over the natural left action of B to Q. The remainder of the proof involves demonstrating that P and Q are projective in the manner specified and that $F = - \underset{A}{\otimes} P$ and $G = - \underset{B}{\otimes} Q$.

3.2 Stable equivalence

From the point of view of representation theory the simplest class of algebras to understand are the semisimple algebras. These algebras have the property that every module is a direct sum of simple modules and so once the simple modules are understood so is the entire module category. Another characterisation of being semisimple is that all modules are projective, thus to gain insight into non-semisimple algebras it makes sense to study the non-projective modules. The setting for this is called the *stable module category*.

We will define the stable module category as a particular quotient of the full module category in such a way that all projective modules are (isomorphic to) zero. For modules X and Y, let $\mathrm{PHom}(X, Y)$ denote the subset of $\mathrm{Hom}(X, Y)$ that consists of all homomorphisms that factor through a projective module. That is $f \in \mathrm{PHom}(X, Y)$

if and only if there is a projective module P and morphisms $g \in \mathrm{Hom}(X,P)$ and $h \in \mathrm{Hom}(P,Y)$ such that the diagram



commutes. We refer to PHom as the projective homomorphisms. The fact that $\mathrm{PHom}(X,Y)$ is actually a subgroup of $\mathrm{Hom}(X,Y)$ should be clear as if f factors through P and g factors through Q then $f+g$ factors through $P \oplus Q$. It is equally clear that if we have $f \in \mathrm{PHom}(X,Y)$ and $g \in \mathrm{Hom}(Y,Z)$ then $gf \in \mathrm{PHom}(X,Z)$, and similarly for composition on the other side.

Definition 3.2 (Stable module category)**.** The *stable module category* of an algebra A, denoted $\underline{\mathrm{Mod}}\,A$, is given by the data:

- **Objects** the same objects as $\mathrm{Mod}\,A$;
- **Morphisms** for modules X and Y, $\underline{\mathrm{Hom}}(X,Y) = \dfrac{\mathrm{Hom}(X,Y)}{\mathrm{PHom}(X,Y)}$

That this description satisfies the requirements for being a category is a consequence of the discussion above. In the stable module category all projective modules are zero objects, in particular all semisimple algebras have equivalent (and zero) stable module categories. The stable module category leads to a natural equivalence relation on algebras called stable equivalence: we say that two algebras are *stably equivalent* if their respective module categories are equivalent. In a similar way to the module category, we will denote by $\underline{\mathrm{mod}}\,A$ the full subcategory of $\underline{\mathrm{Mod}}\,A$ consisting of finitely generated modules.

Example. Let k be a field of characteristic 2. The group algebras for the alternating groups kA_4 and kA_5 are stably equivalent. This can be shown as a consequence of a special case of the Green correspondence for trivial intersection groups.

The Sylow 2-subgroups inside A_5 are isomorphic to the Klein-4 group. If we let $K_4 < A_5$ be the Klein-4 that fixes 5 then it is clear to see that K_4 has the trivial intersection property, that is if $g \in A_5$ then $K_4 \cap gK_4g^{-1}$ is either K_4 or $\{1\}$. The normaliser of K_4 is the alternating group A_4 that fixes 5.

With this set-up the Green correspondence for trivial intersection groups (see [Alp86, theorem 10.1]) tells us that induction and restriction between A_4 and A_5 are inverse operations of one another up to addition of a projective module.

[Alp86] Alperin, *Local representation theory*, Cambridge Studies in Advanced Mathematics, vol. 11, Cambridge University Press, Cambridge, 1986

For the stable module category we would like a characterisation similar to that of theorem 3.1 where we instead think of the isomorphisms as stable isomorphisms. Unfortunately such a result is not true and the best we can do is the following theorem.

Theorem 3.3: *Stable equivalence of Morita type*

Let A and B be R-algebras. If there are bimodules ${}_AP_B$ and ${}_BQ_A$ such that

(a) ${}_AP$, P_B, ${}_BQ$ and Q_A are projective; and

(b) there are projective bimodules ${}_AX_A$ and ${}_BY_B$ and bimodule isomorphisms

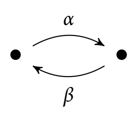
$$ {}_AP \underset{B}{\otimes} Q_A \xrightarrow{\sim} {}_AA_A \oplus {}_AX_A \qquad {}_BQ \underset{A}{\otimes} P_B \xrightarrow{\sim} {}_BB_B \oplus {}_BY_B $$

then we say A and B have a *stable equivalence of Morita type* and in particular they are stably equivalent.

Proof. Mutatis mutandis the proof is the same as that of theorem 3.1. □

Notice that the operations used in the example above to show that kA_4 and kA_5 are stably equivalent over a field of characteristic 2 were that of induction and restriction. Induction from kA_4 to kA_5 is the same as the tensor product with the bimodule ${}_{kA_4}kA_5{}_{kA_5}$ and similarly restriction is the same as the tensor product with ${}_{kA_5}kA_5{}_{kA_4}$. In this way we see that this is also an example of a Morita-type stable equivalence.

We consider an example where two algebras are stably equivalent but this equivalence is not of Morita-type. Let A be the k-algebra for the quiver



$$\alpha\beta = \beta\alpha = 0$$

and B be the k-algebra for the quiver

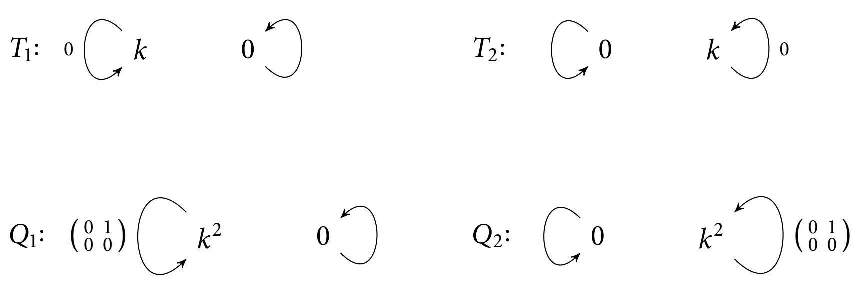
$$\gamma^2 = \delta^2 = 0.$$

The simple modules S_i, and the indecomposable projective modules P_i, for A are

$$S_1\colon\ k \rightleftarrows 0 \qquad\qquad S_2\colon\ 0 \rightleftarrows k$$

$$P_1\colon\ k \underset{0}{\overset{1}{\rightleftarrows}} k \qquad\qquad P_2\colon\ k \underset{1}{\overset{0}{\rightleftarrows}} k$$

whereas for B the simple modules T_i, and the indecomposable projective modules Q_i, are



and these are the only indecomposable modules.

Clearly in both cases when we factor out the projective modules we are simply left with modules for the algebra $k \times k$ and so in particular the stable module categories coincide. If we had a stable equivalence of Morita type then in particular we would have exact functors of the module category

$$F\colon \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, B \qquad\qquad G\colon \mathrm{mod}\, B \longrightarrow \mathrm{mod}\, A$$

satisfying

$$FG(M) = M \oplus (\text{projective})$$

and similarly for GF. If

$$G(T_1) = S_1^m \oplus S_2^n \oplus (\text{projective})$$

then the exact sequences

$$0 \to S_2 \to P_1 \to S_1 \to 0 \qquad\qquad 0 \to S_1 \to P_2 \to S_2 \to 0$$

$$0 \to T_1 \to Q_1 \to T_1 \to 0$$

tell us that

$$\begin{array}{ccccccccc} 0 \longrightarrow & S_2^n \oplus S_1^m \oplus (\text{proj}) & \longrightarrow & P_1^n \oplus P_2^m \oplus (\text{proj}) & \longrightarrow & S_1^n \oplus S_2^m \oplus (\text{proj}) & \longrightarrow 0 \\ & \wr\| & & \wr\| & & \wr\| & \\ & G(T_1) & & G(Q_1) & & G(T_1) & \end{array}$$

is exact and so $m = n$. Since $FG(T_1) = T_1 \oplus (\text{proj})$ then either $F(S_1)$ or $F(S_2)$ is projective: in either case this contradicts $GF(S_i) = S_i \oplus (\text{proj})$.

Though this example demonstrates that the converse to theorem 3.3 is not true, the example is a little contrived and all stable equivalences we shall consider henceforth will be of Morita type.

3.3 Derived equivalence

The final equivalence we wish to mention before introducing separable equivalence is *derived equivalence*. We will skip over most of the details as they are not directly relevant to what is to follow however this section would feel incomplete without at least touching on the subject. For a good reference on derived categories see [Kra07].

[Kra07] Krause, *Derived categories, resolutions, and Brown representability*, Interactions between homotopy theory and algebra, Contemp. Math., vol. 436, Amer. Math. Soc., Providence, RI, 2007, pp. 101–139

In a similar approach to stable equivalence we first define a new type of category derived from the module category and then say that two algebras are derived equivalent if their derived categories are equivalent.

We begin with the category of complexes: in this category objects are cochain complexes of modules

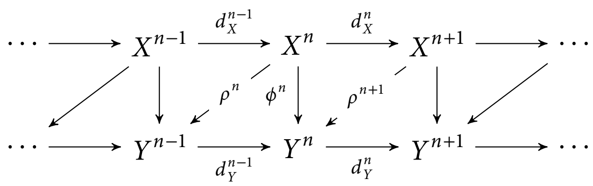
$$\cdots \longrightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \longrightarrow \cdots$$

such that the maps $d^n \circ d^{n-1} = 0$ for all integers n. A morphism between complexes is given by a collection of module morphisms $\phi^n \colon X^n \to Y^n$

$$\begin{array}{ccccccccc} \cdots \longrightarrow & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \phi^{n-1}} & & \downarrow{\scriptstyle \phi^n} & & \downarrow{\scriptstyle \phi^{n+1}} & \\ \cdots \longrightarrow & Y^{n-1} & \xrightarrow[d_Y^{n-1}]{} & Y^n & \xrightarrow[d_Y^n]{} & Y^{n+1} & \longrightarrow \cdots \end{array}$$

such that the squares commute: $d_Y^n \circ \phi^n = \phi^{n+1} \circ d_X^n$.

From the category of complexes we next define the homotopy category. As we factored out the projective homomorphisms in the stable category, here we factor out those morphisms that are null-homotopic. Recall that a chain map ϕ is called null-homotopic if there are maps $\rho^n\colon X^n \to Y^{n-1}$ such that ϕ can be expressed as the sum $\phi^n = d_Y^{n-1} \circ \rho^n + \rho^{n+1} \circ d_X^n$ for all integers n.



Finally we obtain the derived category by formally inverting quasi-isomorphisms. Recall that a morphism of complexes induces a morphism on the cohomology groups. We say that a morphism is a quasi-isomorphism if these induced maps are isomorphisms of the cohomology groups.

That the above procedure does indeed describe a category is beyond the scope of this document, as are many of the properties of the derived category. We simple state that the derived category has the structure of a triangulated category and that two algebras are considered *derived equivalent* if their derived categories are equivalent as triangulated categories. For the relevant details see [Kra07].

[Kra07] Krause, *Derived categories, resolutions, and Brown representability*, Interactions between homotopy theory and algebra, Contemp. Math., vol. 436, Amer. Math. Soc., Providence, RI, 2007, pp. 101–139

The important result for derived equivalence that connects to the theorems of the previous section is the following theorem of Rickard. Note that the original statement of the result is more general than the version stated here.

Theorem 3.4: [Ric91, corollary 5.5]

> If A and B are derived equivalent symmetric algebras over a field k then A and B have a Morita-type stable equivalence.

[Ric91] Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. (2) **43** (1991), no. 1, 37–48

4 Separable equivalence

The preceding chapter introduced several notions of the equivalence of algebras. In each case the existence of a particular pair of bimodules was enough to demonstrate that the algebras were equivalent and it is this idea that we will use to define separable equivalence. This idea was first proposed by Linckelmann in [Lin11b].

[Lin11b] Linckelmann, *Finite generation of Hochschild cohomology of Hecke algebras of finite classical type in characteristic zero*, Bull. Lond. Math. Soc. **43** (2011), no. 5, 871–885

Definition 4.1 (Separable equivalence)**.** Let A and B be R-algebras. We say that A and B are *separably equivalent* if there are bimodules ${}_AM_B$ and ${}_BN_A$ such that

(a) the modules ${}_AM$, M_B, ${}_BN$ and N_A are finitely generated and projective; and

(b) there are bimodules ${}_AX_A$ and ${}_BY_B$ and bimodule isomorphisms

$$ {}_AM \underset{B}{\otimes} N_A \xrightarrow{\sim} {}_AA_A \oplus {}_AX_A \qquad {}_BN \underset{A}{\otimes} M_B \xrightarrow{\sim} {}_BB_B \oplus {}_BY_B $$

Separable equivalence is a generalisation of Morita, derived and stable equivalence of Morita-type. This is easily seen as a direct result of theorems 3.1, 3.3 and 3.4.

The terminology *separable equivalence* comes from the following proposition that was stated by Linckelmann in [Lin11b].

Proposition 4.2. *A finite dimensional algebra A over a field k is separable (in the sense of definition 2.17) if and only if it is separably equivalent to k.*

Proof. Firstly, assume that A is a separable algebra so that A is a summand of $A \underset{k}{\otimes} A$, as A-A-bimodules. Taking $M = {}_AA_k$ and $N = {}_kA_A$, so that $M \underset{k}{\otimes} N = A \underset{k}{\otimes} A$, we have the required isomorphisms.

Now assume that A is separably equivalent to k through bimodules ${}_AM_k$ and ${}_kN_A$. Consider the functor $\mathrm{Hom}_{A-A}(M \underset{k}{\otimes} N, —)$:

$$ \begin{aligned} \mathrm{Hom}_{A-A}\left(M \underset{k}{\otimes} N, —\right) &\cong \mathrm{Hom}_A\left(M, \mathrm{Hom}_A(N, —)\right) \\ &= \mathrm{Hom}_A(M, —) \circ \mathrm{Hom}_A(N, —) \end{aligned} $$

Since M is projective as a left A–module we have that $\operatorname{Hom}_A(M, -)$ is exact. Similarly the functor $\operatorname{Hom}_A(N, -)$ is exact and hence so is the composition. We therefore have that $M \underset{k}{\otimes} N$ is projective as an A-A–bimodule and so A is projective as an A-A–bimodule. □

For a general algebra A over a ring R we would not have that R is a summand of A and so the above proof does not go through. If we had this additional assumption however, (for example if A were free as an R-module) we could follow the same proof.

It will be convenient to talk about situations in which only one of the isomorphisms in the definition of separable equivalence exists. In this situation we will use the language of Bergh and Erdmann (see [BE11, 2506f.]) and say that one algebra *separably divides* the other.

[BE11] Bergh and Erdmann, *The representation dimension of Hecke algebras and symmetric groups*, Adv. Math. **228** (2011), no. 4, 2503–2521

Definition 4.3 (Separably divides)**.** Given two R-algebras A and B, we say that A *separably divides* B if there exists bimodules ${}_AM_B$ and ${}_BN_A$, finitely generated projective on both sides, such that A is a bimodule direct summand of $M \underset{B}{\otimes} N$.

Proposition 4.4. *Let A and B be R-algebras. A and B are separably equivalent if and only if A separably divides B and B separably divides A.*

Proof. One direction is clear, for the other let A separably divide B via (W, X) and B separably divide A via (Y, Z). Let $M = W \oplus Z$ and $N = X \oplus Y$ then the pair (M, N) provide a separable equivalence for A and B. □

The next proposition, which was stated by Linckelmann in [Lin11b], will give us our first example of algebras that are separably equivalent but are not equivalent in any of the more specialised ways we have seen in the preceding chapters.

[Lin11b] Linckelmann, *Finite generation of Hochschild cohomology of Hecke algebras of finite classical type in characteristic zero*, Bull. Lond. Math. Soc. **43** (2011), no. 5, 871–885

Proposition 4.5. *Let G be a finite group, k a field of characteristic $p > 0$. If P is a Sylow p-subgroup of G then kP is separably equivalent to kG.*

Proof. Consider the bimodules ${}_{kP}kG_{kG}$ and ${}_{kG}kG_{kP}$.

Let $\{x_1, x_2, \ldots, x_m\}$ be a set of left coset representatives for P in G. There is an isomorphism of right kP-modules

$$\begin{array}{rcl} kP^m{}_{kP} & \longrightarrow & kG_{kP} \\ (p_1, p_2, \ldots, p_m) & \mapsto & \sum_{i=1}^{m} x_i p_i \end{array}$$

showing that kG_{kP} is projective and similarly for the left module. This isomorphism also demonstrates that as kP-kP–bimodules that kP is a summand of

$$ {}_{kP}kG \underset{kG}{\otimes} kG_{kP} \cong {}_{kP}kG_{kP}. $$

Now consider the reverse tensor product,

$$ {}_{kG}kG \underset{kP}{\otimes} kG_{kG}. $$

Define the map

$$ \begin{aligned} \varphi\colon kG &\longrightarrow {}_{kG}kG \underset{kP}{\otimes} kG_{kG} \\ g &\mapsto \sum_{x \in G/P} gx \otimes x^{-1} \end{aligned} $$

which is a module homomorphism since

$$ \begin{aligned} gh \mapsto &\sum_{x \in G/P} ghx \otimes x^{-1} \\ &= \sum_{x \in G/P} g(hx) \otimes (hx)^{-1}h \\ &= \sum_{y \in G/P} gy \otimes y^{-1}h. \end{aligned} $$

If we compose φ with the homomorphism

$$ \begin{aligned} {}_{kG}kG \underset{kP}{\otimes} kG_{kG} &\longrightarrow kG \\ g \otimes h &\mapsto gh \end{aligned} $$

we have

$$ g \mapsto \sum_{x \in G/P} g = |G : P| g $$

and since P is a Sylow p-subgroup the index $|G : P|$ is invertible and kG is a summand of ${}_{kG}kG \underset{kP}{\otimes} kG_{kG}$. □

For the reader that is familiar with block algebras and defect groups it is important to note that the equivalent statement in this context is also true. If B is a block algebra and D its associated defect group then each module for B is a module for the whole group algebra and so we can restrict from B to kD. On the other hand if we have a module for kD then we can induce to kG and there is a unique summand of the induced module that "belongs" to B. These two functors

$$B \underset{\text{induction and projection}}{\overset{\text{restriction}}{\rightleftarrows}} kD$$

can be written as a tensor product with certain bimodules and it is these bimodules that give the separable equivalence between B and kD.

In the example on page 27 we showed that over a field of characteristic 2, the algebras kA_4 and kA_5 were stably equivalent (and therefore separably equivalent). Notice that proposition 4.5 now tells us that in addition these algebras are separably equivalent to kK_4.

We now present some new results relating to separable equivalence.

Proposition 4.6. *Let A, B and C be algebras over a field k. If A separably divides B then $A \underset{k}{\otimes} C$ separably divides $B \underset{k}{\otimes} C$.*

Proof. Let ${}_AM_B$ and ${}_BN_A$ be a pair of bimodules with the property that

$${}_AM \underset{B}{\otimes} N_A \cong {}_AA_A \oplus {}_AX_A.$$

The tensor product $M \underset{k}{\otimes} C$ is an $(A \underset{k}{\otimes} C)$-$(B \underset{k}{\otimes} C)$–bimodule with the actions

$$(a \otimes c_1)(m \otimes c_2)(b \otimes c_3) = amb \otimes c_1c_2c_3$$

and we can similarly define a $(B \underset{k}{\otimes} C)$-$(A \underset{k}{\otimes} C)$ action on $N \underset{k}{\otimes} C$.

Thus we have that

$$\begin{aligned}(M \underset{k}{\otimes} C) \underset{B \otimes C}{\otimes} (N \underset{k}{\otimes} C) &\cong (M \underset{B}{\otimes} N) \underset{k}{\otimes} C \\ &\cong (A \oplus X) \underset{k}{\otimes} C \\ &\cong (A \underset{k}{\otimes} C) \oplus (X \underset{k}{\otimes} C)\end{aligned}$$

Now if ${}_AM$ is projective then ${}_AM$ is a summand of A^n for some n. Therefore $M \underset{k}{\otimes} C$ is a summand of

$$A^n \underset{k}{\otimes} C \cong (A \underset{k}{\otimes} C)^n$$

and hence is projective. □

In the proof of proposition 4.5 the bimodules that we used to form the separable equivalence were duals of one another. In particular in this situation we have that tensoring with M is both left and right adjoint to tensoring with N. This leads us to a new definition which we call symmetrical separable equivalence.

Definition 4.7 (Symmetrical separable equivalence)**.** Let A and B be finite dimensional algebras. We say that A and B are *symmetrically separably equivalent* if there is a separable equivalence (M, N) such that $- \otimes M$ is both left and right adjoint to $- \otimes N$.

Proposition 4.8. *If A and B are symmetric algebras then A and B are separably equivalent if and only if A and B are symmetrically separably equivalent.*

Proof. Let (M, N) be a separable equivalence for A and B. Consider the modules $M \oplus N^*$ and $N \oplus M^*$. It is clear that these two modules satisfy the summand properties of separable equivalence. We wish to show that the tensor functors are adjoints of one another so let X be an A-module and Y be a B-module.

First note that since

$$\mathrm{Hom}_B(B, Y) \cong Y \underset{B}{\otimes} \mathrm{Hom}_B(B, B)$$

and M is projective (and therefore a direct sum of summands of B) we also have

$$\mathrm{Hom}_B(M, Y) \cong Y \underset{B}{\otimes} \mathrm{Hom}_B(M, B)$$

(cf. [AF92, prop 20.10]). Thus we have

[AF92] Anderson and Fuller, *Rings and categories of modules*, second ed., Graduate Texts in Mathematics, vol. 13, Springer-Verlag, New York, 1992

$$\begin{aligned}
\mathrm{Hom}_B(X \underset{A}{\otimes} M, Y) &\cong \mathrm{Hom}_A(X, \mathrm{Hom}_B(M, Y)) && \text{via the tensor-Hom adjunction} \\
&\cong \mathrm{Hom}_A(X, Y \underset{B}{\otimes} \mathrm{Hom}(M, B)) && \text{by the argument above} \\
&\cong \mathrm{Hom}_A(X, Y \underset{B}{\otimes} M^*) && \text{due to symmetry of } B
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\mathrm{Hom}_A(Y \underset{B}{\otimes} M^*, X) &\cong \mathrm{Hom}_B(Y, \mathrm{Hom}_A(M^*, X)) \\
&\cong \mathrm{Hom}_B(Y, X \underset{A}{\otimes} M^{**}) \\
&\cong \mathrm{Hom}_B(Y, X \underset{A}{\otimes} M) && \text{(cf. [AF92, prop 20.17])}
\end{aligned}$$

That M^* is projective as a left and right module follows simply from the adjunction.

□

4.1 Complexity

There are certain properties of algebras that are preserved through separable equivalence. One such property is the complexity of an algebra. Here we introduce what is meant by the complexity of a module and of an algebra and go on to prove that it is unchanged by separable equivalence. This result seems to be well-known but we don't know of any complete statement or proof in print.

Definition 4.9 (Complexity)**.** Let A be an algebra over a field k, let M be an A-module and

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

a projective resolution of M, which we will denote by P_*.

If there exists an integer d, such that for some $\lambda \in \mathbb{N}$ we have $\dim(P_n) \leq \lambda n^{d-1}$ for all $n \in \mathbb{N}$ then we say that P_* has *finite complexity* and we call the smallest such d the *complexity of the resolution*, which we denote by $\operatorname{cx} P_*$.

The *complexity of the module M* is equal to the complexity of a minimal projective resolution of M.

The *complexity of an algebra* is the maximal complexity for a module of that algebra.

In order to show that the definition of complexity of a module makes sense we have the following lemma. This shows that an equivalent definition would be the minimum complexity for any projective resolution of M.

Lemma 4.10. *If we have a minimal projective resolution*

$$\cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

and any projective resolution

$$\cdots \longrightarrow Q_1 \xrightarrow{e_1} Q_0 \xrightarrow{e_0} M \longrightarrow 0$$

then $\operatorname{cx}(P_*) \leq \operatorname{cx}(Q_*)$.

Proof. Since P_* is a minimal projective resolution, theorem 2.11 tells us that

$$Q_i \cong P_i \oplus Q_i'$$

for some projectives Q_i' in such a way that $\ker(e_i) = \ker(d_i) \oplus Q_i'$ and $e_i|_{P_i} = d_i$. Thus $\dim(P_i) \leq \dim(Q_i)$ for all i and $\operatorname{cx}(P_*) \leq \operatorname{cx}(Q_*)$. □

We require two further lemmas before we can prove the main result of this section: that separable equivalence preserves complexity.

Lemma 4.11. *Let A and B be finite dimensional algebras over a field k and ${}_AM_B$ a bimodule that is finitely generated projective as a both an A-module and as a B-module. If we have a projective resolution of A-modules*

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

then $\operatorname{cx}(X \underset{A}{\otimes} M) \leq \operatorname{cx}(X) \leq \operatorname{cx}(P_*)$.

Proof. Since M is projective as a A-module the functor

$$- \underset{A}{\otimes} M\colon \operatorname{mod} A \longrightarrow \operatorname{mod} B$$

is exact and since M is projective as a B-module each $P_i \underset{A}{\otimes} M$ is projective. We therefore have that $P_* \underset{A}{\otimes} M$ is a projective resolution of $X \underset{A}{\otimes} M$.

It is clear that $\dim(P_i \underset{A}{\otimes} M) \leq \dim(P_i \underset{k}{\otimes} M)$ and since M and B are both finitely generated $\dim(P_i \underset{k}{\otimes} M) = \dim(P_i)\dim(M) < \infty$.

Finally if $\dim(P_i) \leq f(i)$ for some polynomial f then $\dim(P_i \underset{A}{\otimes} M) \leq \dim(M) f(i)$ and hence $\operatorname{cx}(P_* \underset{A}{\otimes} M) \leq \operatorname{cx}(P_*)$. □

Lemma 4.12. *If A separably divides B via the modules $({}_AM_B, {}_BN_A)$ and X_A is an A-module then* $\operatorname{cx}(X \underset{A}{\otimes} M) = \operatorname{cx}(X)$.

Proof. It suffices to show that $\operatorname{cx}(X) \leq \operatorname{cx}(X \underset{A}{\otimes} M \underset{B}{\otimes} N)$ as together with lemma 4.11 this gives

$$\operatorname{cx}(X \underset{A}{\otimes} M \underset{B}{\otimes} N) \leq \operatorname{cx}(X \underset{A}{\otimes} M) \leq \operatorname{cx}(X) \leq \operatorname{cx}(X \underset{A}{\otimes} M \underset{B}{\otimes} N)$$

and we will have equality throughout. Since $X \underset{A}{\otimes} M \underset{B}{\otimes} N \cong X \oplus X'$ for some X' we have that a minimal projective resolution of $X \underset{A}{\otimes} M \underset{B}{\otimes} N$ is simply the direct sum of the minimal projective resolutions of X and X' and thus the given inequality is immediate. □

The next theorem is a direct consequence of the previous three lemmas.

Theorem 4.13:

> If A separably divides B then $\mathrm{cx}(A) \leq \mathrm{cx}(B)$.
>
> If A and B are separably equivalent then $\mathrm{cx}(A) = \mathrm{cx}(B)$.

Let G be a finite group and k a field of characteristic p. If V and W are modules for the group algebra kG we may form the tensor product $V \otimes W$ where the action

$$(v \otimes w)g = vg \otimes wg.$$

Note that this differs from the tensor product over an algebra we have hitherto used. With this diagonal action we have the property that if P is a projective module and V is any module then $P \otimes V$ is projective (see for instance [Alp86, lemma 7.4]). If we take a minimal projective resolution of the trivial module k

[Alp86] Alperin, *Local representation theory*, Cambridge Studies in Advanced Mathematics, vol. 11, Cambridge University Press, Cambridge, 1986

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k \longrightarrow 0$$

then we can form a projective resolution of a module V via

$$\begin{array}{c} \cdots \longrightarrow P_1 \otimes V \longrightarrow P_0 \otimes V \longrightarrow k \otimes V \longrightarrow 0 \\ \hphantom{\cdots \longrightarrow P_1 \otimes V \longrightarrow P_0 \otimes V \longrightarrow} \cong \hphantom{\longrightarrow 0} \\ \hphantom{\cdots \longrightarrow P_1 \otimes V \longrightarrow P_0 \otimes V \longrightarrow} V \hphantom{\longrightarrow 0} \end{array}$$

This demonstrates that the complexity of a module for a group algebra is always bounded above by the complexity of the trivial module.

The following theorem of Alperin and Evens tells us that the complexity of a module for a group algebra can be calculated on elementary p-subgroups.

[AE81] Alperin and Evens, *Representations, resolutions and Quillen's dimension theorem*, J. Pure Appl. Algebra **22** (1981), no. 1, 1–9

Theorem 4.14: [AE81]

> Let G be a finite group and k a field of prime characteristic p. If V is a kG-module then
>
> $$\mathrm{cx}_{kG}(V) = \max_E \left(\mathrm{cx}_E(V\downarrow_E) \right)$$
>
> where E runs through all elementary abelian subgroups of G and $V\downarrow_E$ denotes the restriction of V to the subgroup E.

Let k be a field of characteristic p and C_{p^n} be the cyclic group of order p^n. Theorem 4.14 tells us that the complexity of kC_{p^n} is the same as the complexity of kC_p. If we use the isomorphism $kC_p \cong \frac{k[x]}{(x^p)}$ then

$$\cdots \xrightarrow{x} \frac{k[x]}{(x^p)} \xrightarrow{x^{p-1}} \frac{k[x]}{(x^p)} \xrightarrow{x} \frac{k[x]}{(x^p)} \longrightarrow k \longrightarrow 0$$

is a minimal projective resolution of the trivial module and hence the complexity of this module is 1. As a straightforward consequence of the horseshoe lemma we can see that the complexity of any module is the maximum complexity amongst its composition factors and thus we have shown that the complexity of any cyclic group is also 1.

As we have seen that separable equivalence preserves complexity, the discussion above leads us naturally to ask if any algebras for cyclic p-groups are separably equivalent. A partial answer to this question will be presented in section 5.6.

4.2 Categorical formulation

The notion of separable equivalence we have used thus far was presented in terms of bimodules for two algebras but note that we could have just as easily defined this equivalence in terms of functors between the module categories. We make this clear with the following theorem.

Theorem 4.15: *Separable equivalence*

The algebras A and B are separably equivalent if and only if there are exact functors F and G

$$\operatorname{mod} A \underset{G}{\overset{F}{\rightleftarrows}} \operatorname{mod} B$$

such that

(a) if P is a projective A-module then FP is a projective B-module;

(b) if Q is a projective B-module then GQ is a projective A-module;

(c) for some functor $S\colon \operatorname{mod} A \to \operatorname{mod} A$ there is an equivalence of functors

$$GF \xrightarrow{\sim} \mathrm{Id}_{\operatorname{mod} A} \oplus S$$

(d) for some functor $T\colon \operatorname{mod} B \to \operatorname{mod} B$ there is an equivalence of functors

$$FG \xrightarrow{\sim} \mathrm{Id}_{\operatorname{mod} B} \oplus T$$

Proof. Let A and B be separably equivalent via the modules ${}_AM_B$ and ${}_BN_A$ and let

$$F = - \underset{A}{\otimes} M \qquad\qquad G = - \underset{B}{\otimes} N.$$

The composition GF is given by

$$\begin{aligned} GF = - \underset{A}{\otimes} M \underset{B}{\otimes} N &\cong (- \underset{A}{\otimes} A) \oplus (- \underset{A}{\otimes} X) \\ &\cong \mathrm{Id}_{\mathrm{mod}\,A} \oplus (- \underset{A}{\otimes} X) \end{aligned}$$

for some A-A–bimodule X and similarly for the composition FG.

That F is exact is a direct consequence of ${}_AM$ being projective. Finally if P_A is a projective A-module then

$$\begin{aligned} \mathrm{Hom}_B(FP, -) = \mathrm{Hom}_B(P \underset{A}{\otimes} M, -) &\cong \mathrm{Hom}_A\left(P, \mathrm{Hom}_B(M, -)\right) \\ &\cong \mathrm{Hom}_A(P, -) \circ \mathrm{Hom}_B(M, -). \end{aligned}$$

As both P_A and M_B are projective the composition is exact and hence FP is projective. Again we may make corresponding arguments for the functor G.

For the opposite implication assume that we have the functors F and G described in the theorem and $M_B = FA$ and $N_A = GB$.

We can carry the natural left action of A on itself over to the module M. If λ_a is the endomorphism of A_A representing left multiplication by a then $F(\lambda_a)$ gives an endomorphism of M_B. Thus we can define the action of A on M via

$$\begin{array}{rcl} A & \longrightarrow & \mathrm{End}(M_B) \\ a & \mapsto & F(\lambda_a) \end{array}$$

In this way we obtain bimodules ${}_AM_B$ and ${}_BN_A$ and since F and G are right exact the Eilenberg–Watts theorem ([Eil60]) tells us that

[Eil60] Eilenberg, *Abstract description of some basic functors*, J. Indian Math. Soc. (N.S.) **24** (1960), 231–234 (1961)

$$F \cong - \underset{A}{\otimes} M \qquad\qquad G \cong - \underset{B}{\otimes} N.$$

We need now only show that M is left and right projective. Right projectivity is immediate from the definition of F and from the fact that A is right projective. For left projectivity let $f : X \to FA$ be any surjective A-homomorphism. We need to demonstrate that f has a right inverse.

Since A is a summand of GFA and A is projective we have a map g making the following diagram commute

$$\begin{array}{ccccc} & & A & & \\ & \overset{\exists g}{\swarrow} & \downarrow & & \\ GX & \xrightarrow[Gf]{} & A \oplus A' & \longrightarrow & 0 \\ & & \wr\| & & \\ & & GFA & & \end{array}$$

with the bottom row exact. After applying F to this we get

$$\begin{array}{ccccc} & & FA & & \\ & \overset{Fg \cong \binom{h}{h'}}{\swarrow} & \downarrow & & \\ X \oplus X' & \xrightarrow[FGf \cong \left(\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right)]{} & FA \oplus FA' & \longrightarrow & 0 \\ \wr\| & & & & \\ FGX & & & & \end{array}$$

and hence h is the required section for f. □

We may take theorem 4.15 as a more generalised definition of separable equivalence. The advantage of the categorical definition is that we no longer need to refer to module categories and can talk about separable equivalence of general exact categories. In a similar way we may generalise the definition of symmetric separable equivalence and in this case we may even drop the requirement that the categories are exact.

Definition 4.16 (Separable equivalence)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be exact categories. We say that these categories are separably equivalent if there exists exact functors

$$\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

such that

(a) if P is a projective object of $\mathcal{A}$ then FP is a projective object of $\mathcal{B}$;

(b) if Q is a projective object of $\mathcal{B}$ then GQ is a projective object of $\mathcal{A}$;

(c) the identity functor on $\mathcal{A}$ is a summand of GF;

(d) the identity functor on $\mathcal{B}$ is a summand of FG.

Definition 4.17 (Symmetrical separable equivalence)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. We say that these categories are *symmetrically separably equivalent* if there are functors

$$\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

such that

(a) (F, G) is an adjoint pair;

(b) (G, F) is an adjoint pair;

(c) the identity functor on $\mathcal{A}$ is a summand of GF;

(d) the identity functor on $\mathcal{B}$ is a summand of FG.

Remark. If $\mathcal{A}$ and $\mathcal{B}$ are module categories then the adjointness implies that the functors are exact and that projective modules are sent to projective modules.

We wish to demonstrate that given a separable equivalence between algebras this generates further separable equivalences between other more general categories. We will restrict in what follows to the case of symmetrical separable equivalence and thus we begin with some properties of adjointness.

Lemma 4.18. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be k-categories. Given functors L and R*

$$\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{B}$$

we can define in a natural way the functors

$$\mathrm{Fun}(\mathcal{B}, \mathcal{C}) \underset{- \circ R}{\overset{- \circ L}{\rightleftarrows}} \mathrm{Fun}(\mathcal{A}, \mathcal{C}) \qquad \mathrm{Fun}(\mathcal{C}, \mathcal{A}) \underset{R \circ -}{\overset{L \circ -}{\rightleftarrows}} \mathrm{Fun}(\mathcal{C}, \mathcal{B})$$

If (L, R) is an adjoint pair then $(—R, —L)$ and $(L—, R—)$ are adjoint pairs.

Proof. We prove that $(-R, -L)$ is an adjunction which should be sufficient to demonstrate the argument for the other pair.

Let $\varepsilon: 1 \Rightarrow RL$ and $\eta: LR \Rightarrow 1$ be the unit and counit of the adjunction.

For each $X: \mathcal{A} \to \mathcal{C}$ and $Y: \mathcal{B} \to \mathcal{C}$ define the natural transformations

$$\begin{array}{ccc} \mathrm{Nat}(XR, Y) & \xrightarrow{\Phi_{X,Y}} & \mathrm{Nat}(X, YL) \\ \xi & \mapsto & \xi L \circ X\varepsilon \end{array}$$

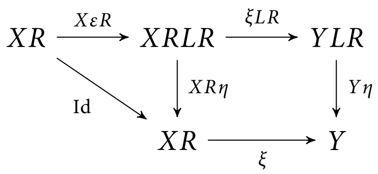
and

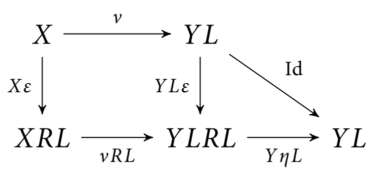
$$\begin{array}{ccc} \mathrm{Nat}(X, YL) & \xrightarrow{\Psi_{X,Y}} & \mathrm{Nat}(XR, Y) \\ \nu & \mapsto & Y\eta \circ \nu R \end{array}$$

We show that Φ and Ψ are inverses of one another:

$$\begin{aligned} \Psi_{X,Y}\Phi_{X,Y}(\xi) &= Y\eta \circ \Phi_{X,Y}(\xi)R \\ &= Y\eta \circ \xi LR \circ X\varepsilon R \\ &= \xi \circ XR\eta \circ X\varepsilon R \\ &= \xi \end{aligned} \qquad \begin{aligned} \Phi_{X,Y}\Psi_{X,Y}(\nu) &= \Psi_{X,Y}(\nu)L \circ X\varepsilon \\ &= Y\eta L \circ \nu RL \circ X\varepsilon \\ &= Y\eta L \circ YL\varepsilon \circ \nu \\ &= \nu \end{aligned}$$

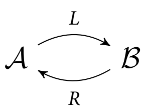
The diagrams

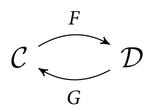




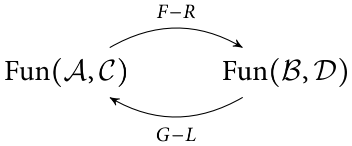
should make this clear. ☐

Lemma 4.19. *More generally, let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be k-categories. Given functors*





we can define the functors



If (L, R) and (F, G) are both adjoint pairs then so is $(F - R, G - L)$.

Proof. The proof is similar to that of lemma 4.18. The mappings are given by

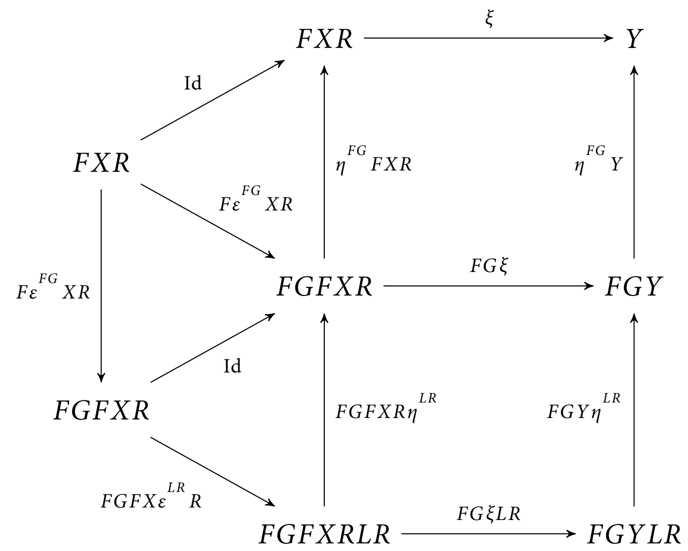
$$\begin{array}{ccc} \mathrm{Nat}(FXR, Y) & \xrightarrow{\Phi_{X,Y}} & \mathrm{Nat}(X, GYL) \\ \xi & \mapsto & G\xi L \circ \varepsilon^{FG} X \varepsilon^{LR} \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Nat}(X, GYL) & \xrightarrow{\Psi_{X,Y}} & \mathrm{Nat}(FXR, Y) \\ \nu & \mapsto & \eta^{FG} Y \eta^{LR} \circ F\nu R \end{array}$$

where ε and η are the units and counits of the adjunctions with superscripts indicating the adjunction in question.

One direction of the proof that these are inverse mappings is demonstrated in the following commutative diagram.



□

The general definition of separable equivalence together with the lemmas above give us the following theorem that provides separable equivalences between functor categories from those of algebras, the proof of which is a direct consequence.

Proposition 4.20. *Let A and B be a pair of symmetrically separably equivalent k-algebras. Let C and D be a second pair of symmetrically separably equivalent k-algebras and let $\mathcal{E}$ be a small k-category. We have the following symmetrical separable equivalences of functor categories:*

(a) $\mathrm{Fun}(\mathrm{mod}\, A, \mathrm{mod}\, C) \sim \mathrm{Fun}(\mathrm{mod}\, B, \mathrm{mod}\, D)$,

(b) $\mathrm{Fun}(\mathrm{mod}\, A, \mathcal{E}) \sim \mathrm{Fun}(\mathrm{mod}\, B, \mathcal{E})$,

(c) $\mathrm{Fun}(\mathcal{E}, \mathrm{mod}\, A) \sim \mathrm{Fun}(\mathcal{E}, \mathrm{mod}\, B)$.

If we consider the restriction of the functors that make up a separable equivalence we may find that some subcategories are also equivalent. For instance if we have a symmetrical separable equivalence (F, G) between two categories $\mathcal{A}$ and $\mathcal{B}$, such that (F, G) restrict to functors between full subcategories $\mathcal{A}'$ and $\mathcal{B}'$ then in fact we have a symmetrical separable equivalence of these subcategories. This follows simply from the fact that the identity functor on the subcategory is the restriction of that of the parent category and additionally that, for full subcategories, the Hom-sets are equal to those of the parent category. This together with a dual result gives us:

Proposition 4.21. *Let $\mathcal{A}$ and $\mathcal{B}$ be symmetrically separably equivalent categories via (F, G). If we have full subcategories $\mathcal{A}' < \mathcal{A}$ and $\mathcal{B}' < \mathcal{B}$ such that $F\mathcal{A}' \subseteq \mathcal{B}'$ and $G\mathcal{B}' \subseteq \mathcal{A}'$ then we have the following symmetrical separable equivalences:*

(a) $\mathcal{A}' \sim \mathcal{B}'$;

(b) $\mathcal{A}/\mathcal{A}' \sim \mathcal{B}/\mathcal{B}'$;

(c) $\mathrm{Fun}\left(\mathcal{A}/\mathcal{A}', \mathrm{mod}\, k\right) \sim \mathrm{Fun}\left(\mathcal{B}/\mathcal{B}', \mathrm{mod}\, k\right)$.

Proof. The proof of (a) is clear from the discussion above and (b) from a dual argument. For (c) we need only note that

$$\mathrm{Fun}\left(\mathcal{A}/\mathcal{A}', \mathrm{mod}\, k\right)$$

is a full subcategory of $\mathrm{Fun}(\mathcal{A}, \mathrm{mod}\, k)$ via the embedding that composes a functor with the obvious projection

$$\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{A}'$$

and that the equivalence given by proposition 4.20 restricts to functors of these subcategories. □

Example. If A and B are separably equivalent symmetric algebras then the subcategories of projective modules, $\mathrm{proj}\, A$ and $\mathrm{proj}\, B$, are symmetrically separably equivalent and the category of representations of their stable categories $\mathrm{Fun}(\underline{\mathrm{mod}}\, A, \mathrm{mod}\, k)$ and $\mathrm{Fun}(\underline{\mathrm{mod}}\, B, \mathrm{mod}\, k)$ are symmetrically separably equivalent.

5 Representation type

In this chapter we will see another property of an algebra that is preserved by separable equivalence: the representation type of the algebra. We begin with definitions for finite, tame and wild representation types and then state Drozd's famous theorem that these are the only possibilities. The proof of this theorem is far beyond the scope of this text but for the interested reader full details can found in [Dro80] or [CB88].

[Dro80] Drozd, *Tame and wild matrix problems*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math., vol. 832, Springer, Berlin, 1980, pp. 242–258

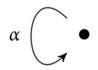
[CB88] Crawley-Boevey, *On tame algebras and bocses*, Proc. London Math. Soc. (3) **56** (1988), no. 3, 451–483

In the remainder of the text we will assume that all algebras are finite dimensional over an algebraically closed field.

5.1 Finite representation type

Definition 5.1 (Finite representation type)**.** An algebra A is said to have *finite representation type* if there exists only finitely many isomorphism classes of indecomposable right (equivalently left) A-modules.

Example. We have already seen several examples of algebras of finite representation type. For instance the path algebra of



$$\alpha^2 = 0$$

has two indecomposable modules that can be represented by:

$$k \qquad \begin{array}{c} k \\ \downarrow \\ k \end{array}$$

The first module is the one-dimensional simple module where α acts as zero. The second is a two-dimensional module where the arrow represents the action of α. Thus if e_1 and e_2 are basis elements then α acts by

$$\begin{aligned} e_1 &\mapsto e_2 \\ e_2 &\mapsto 0 \end{aligned}$$

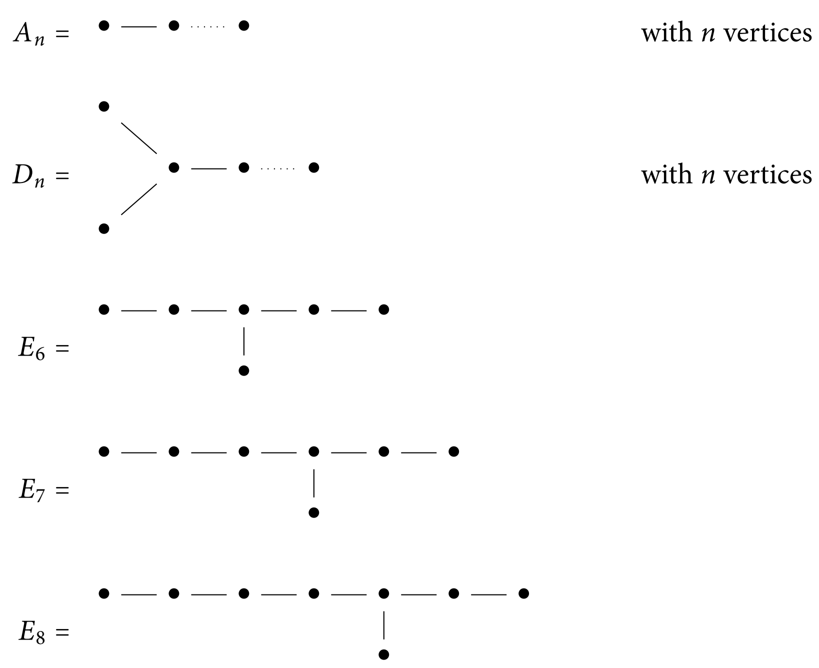
The algebras of finite representation type are very well studied and in fact for group algebras they are completely classified. If the characteristic of k does not divide the order of a group G then kG is semisimple and hence of finite-type. If the characteristic divides the order of G however, then Higman proved in [Hig54] that kG is of finite-type if and only if a Sylow p-subgroup is cyclic. More generally if a block algebra is of finite type then it has cyclic defect group.

A further classification of finite-type algebras comes from Gabriel's theorem (a good reference text for this theorem can be found in [BGP73, 3.1] or [ASS06, VII 5.10]). If Q is a finite connected quiver without oriented cycles then the path algebra kQ is of finite representation type if and only if the undirected version of Q is one of the Dynkin diagrams

[Hig54] Higman, *Indecomposable representations at characteristic p*, Duke Math. J. **21** (1954), 377–381

[BGP73] Bernšteĭn, Gel′fand, and Ponomarev, *Coxeter functors, and Gabriel's theorem*, Uspehi Mat. Nauk **28** (1973), no. 2(170), 19–33

[ASS06] Assem, Simson, and Skowroński, *Elements of the representation theory of associative algebras. Vol. 1*, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006,



Together with theorem 1.7 of chapter VII of [ASS06] this fully classifies the indecomposable finite dimensional hereditary algebras up to Morita equivalence. Note that hereditary algebras are a particular nice collection of algebras characterised by the property that submodules of projective modules are themselves projective.

5.2 Tame representation type

Definition 5.2 (Tame representation type)**.** An algebra A over a field k is said to have *tame representation type* if it does not have finite representation type and given any $d \in \mathbb{N}$ there is a finite set of $k[t]$-A–bimodules $\{X_i\}$, free and finitely-generated as $k[t]$-modules, such that for all but finitely many d-dimensional indecomposable A-modules M (up to isomorphism) we have

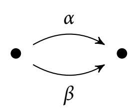
$$M \cong \frac{k[t]}{(t-\lambda)} \underset{k[t]}{\otimes} X_i$$

for some X_i and some $\lambda \in k$.

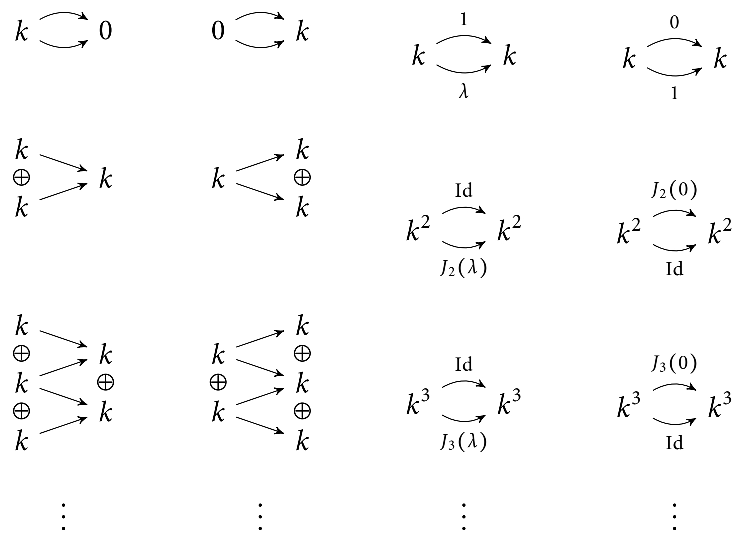
Remark. Some authors include finite representation type within the class of tame representation type however here we consider the two to be mutually exclusive.

The definition of tame given above is simply a precise way of saying that all but finitely many (isomorphism classes of) indecomposable A-modules of a given dimension are covered by a finite set of 1-parameter families of modules.

Example. Consider the path algebra for the quiver



This has the following isomorphism classes of indecomposable modules:



where: $J_n(\lambda)$ is the $n \times n$ Jordan block with eigenvalue $\lambda \in k$; unmarked arrows $k \to k$, represent the identity morphism; the notation $\searrow$ represents the arrow α; and the notation $\nearrow$ represents the arrow β.

Thus the algebra is tame: the third column contains 1-parameter families of modules and all the other columns contain the finite number of exceptions.

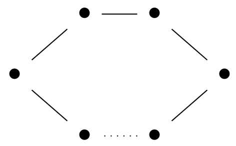
[BD77] Bondarenko and Drozd, *The representation type of finite groups*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **71** (1977), 24–41, 282,

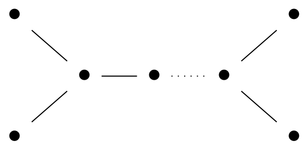
[Rin80] Ringel, *The rational invariants of the tame quivers*, Invent. Math. **58** (1980), no. 3, 217–239

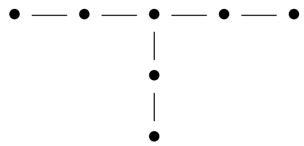
There has been plenty of work on classifying the tame algebras. For instance Bondarenko and Drozd showed in [BD77] that the only tame group algebras are those with dihedral, semidihedral, or generalized quaternion Sylow 2-subgroups in characteristic 2.

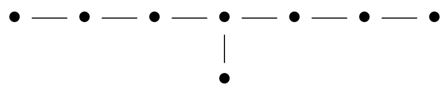
We saw earlier a classification of the path algebras of finite connected quivers that are of finite-type and a similar classification exists for the tame algebras (see [Rin80]). The path algebra of a finite connected quiver is tame if and only if the quiver has no oriented cycles and the undirected version of the quiver is one of the following

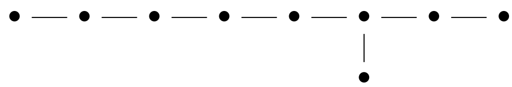
Euclidean diagrams.

$\tilde{A}_n =$  with $n+1$ vertices

$\tilde{D}_n =$  with $n+1$ vertices

$\tilde{E}_6 =$ 

$\tilde{E}_7 =$ 

$\tilde{E}_8 =$ 

Notice that the example of an algebra of tame representation type we saw above is an algebra for a quiver of type $\tilde{A}_1$.

5.3 Wild representation type

Definition 5.3 (Wild representation type). An algebra A over a field k is said to have *wild representation type* if there is a $k\langle u,v\rangle$-A–bimodule X, finitely generated free as a $k\langle u,v\rangle$-module, such that

- if M is an indecomposable right $k\langle u,v\rangle$-module then $M \underset{k\langle u,v\rangle}{\otimes} X$ is indecomposable;
- for $k\langle u,v\rangle$-modules M and N: if $M \underset{k\langle u,v\rangle}{\otimes} X \cong N \underset{k\langle u,v\rangle}{\otimes} X$ then $M \cong N$.

Remark. The notation $k\langle u, v\rangle$ represents the free k-algebra on two generators.

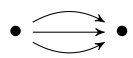
One way of interpreting this definition is that an algebra has wild representation type if its module category is at least as complicated as that of the free algebra on two generators. A d-dimensional module of this algebra corresponds to a pair of $d \times d$ matrices. Two modules are then isomorphic if and only if there is a change of basis sending one pair of matrices to the other. Thus classifying modules for the free algebra on two generators is equivalent to classifying pairs of matrices up to simultaneous conjugation, which is a notoriously hard problem.

Theorem 5.4: *Drozd, 1977*

> Every finite-dimensional algebra over an algebraically closed field has exactly one representation type: finite, tame or wild.

Drozd's trichotomy theorem tells us that if an algebra is not finite or tame then it must be wild. For example, all group algebras that we have not already classified as finite-type or tame-type must be wild. This means that the vast majority of group algebras are wild: all groups except those with dihedral, semidihedral or generalized quaternion Sylow 2-subgroups in characteristic 2 (tame type), or cyclic groups in any characteristic (finite type). Similarly the classification of path algebras for quivers without directed cycles tells us that any other quiver than those of the Dynkin or Euclidean types must be wild.

Example. Let A be the path algebra for the quiver



over a field k and consider the $k\langle u, v\rangle$-A–bimodule

$$X = \; k\langle u,v\rangle \; \overset{u}{\underset{v}{\xrightarrow{\;1\;}}} \; k\langle u,v\rangle$$

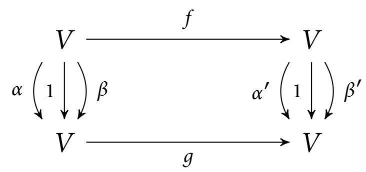
A module for $k\langle u, v\rangle$ is given by a k-vector space, V and two endomorphisms α and β. Given such a module $V_{\alpha,\beta}$ we have that

$$V_{\alpha,\beta} \underset{k\langle u,v\rangle}{\otimes} X \;=\; V \overset{\alpha}{\underset{\beta}{\xrightarrow{\;1\;}}} V$$

which is indecomposable if $V_{\alpha,\beta}$ is indecomposable. Moreover if

$$V_{\alpha,\beta} \underset{k\langle u,v\rangle}{\otimes} X \cong V_{\alpha',\beta'} \underset{k\langle u,v\rangle}{\otimes} X$$

then we have a commutative diagram with isomorphisms f and g



We must have that $f = g$ by commutativity of the identity square. We also have

$$f\alpha = \alpha' f \qquad\qquad f\beta = \beta' f$$

by the other two squares and so $V_{\alpha,\beta} \cong V_{\alpha',\beta'}$. This is enough to show that X satisfies the properties of definition 5.3 and hence A is wild.

5.4 Preservation of representation type

Linckelmann showed in proposition 3.3 of [Lin11b] that if two symmetric algebras are separably equivalent then they must have the same representation type. We aim to show that this result holds for general finite dimensional algebras.

[Lin11b] Linckelmann, *Finite generation of Hochschild cohomology of Hecke algebras of finite classical type in characteristic zero*, Bull. Lond. Math. Soc. **43** (2011), no. 5, 871–885

Proposition 5.5. *Let A and B be k-algebras such that A separably divides B. If B has finite representation type then so does A. Moreover if B is semisimple then so is A.*

Proof. Let ${}_AM_B$ and ${}_BN_A$ be bimodules giving the separable division so that ${}_AA_A$ is a bimodule summand of ${}_AM \underset{B}{\otimes} N_A$.

Let $X_1, \dots, X_r$ be the indecomposable B-modules.

Let $Y_1, \dots, Y_s$ be the indecomposable summands of the A-modules $X_i \underset{B}{\otimes} N$:

$$\{Y_1, \dots, Y_s\} = \left\{ X \,\middle|\, X \text{ is a summand of } X_i \underset{B}{\otimes} N \text{ for some } 1 \le i \le r \right\}$$

Let Y be an indecomposable A-module. We will show that Y is isomorphic to one of the modules Y_j and hence we have only finitely many isomorphism classes of indecomposable A-modules.

Since A is a summand of $M \underset{B}{\otimes} N$ we know that Y is a summand of $Y \underset{A}{\otimes} M \underset{B}{\otimes} N$. This means that there is an indecomposable B-module X_i such that Y is a summand of $X_i \underset{B}{\otimes} N$ and hence isomorphic to some Y_j.

If B is semisimple then the same proof goes through with the additional property that each X_i is projective. This implies that each Y_j is also projective and hence A is semisimple. □

There is an analogous result to proposition 5.5 for tame representation type, but the proof requires an interpretation of the space of modules as a collection of affine varieties. For tame algebras, each 1-parameter family of modules will correspond to a 1-dimensional variety. In the proof of the proposition we will draw a contradiction by showing that if a separable factor is of wild type then one of these varieties must have dimension greater than 1.

Let A be a k-algebra with k-basis $\{1 = a_1, \ldots, a_s\}$ so that

$$a_i a_j = \sum_{l=1}^{s} \lambda_l^{(ij)} a_l$$

for some constants $\lambda_l^{(ij)} \in k$.

Now each d-dimensional A-module M corresponds to a ring homomorphism

$$A \longrightarrow \mathrm{End}_k(k^d)$$

so there are $d \times d$ matrices α_l, $l \in \{1, \ldots, s\}$ defining M. We can thus associate M with the element $\alpha = (\alpha_1, \ldots, \alpha_s) \in k^{sd^2}$.

Conversely, a point $\alpha \in k^{sd^2}$ corresponds to an A-module if the equations

$$\alpha_1 = \mathrm{Id}_d$$

$$\alpha_i \alpha_j = \sum_{l=1}^{s} \lambda_l^{(ij)} \alpha_l$$

are satisfied. Since the only requirements on α are a series of linear equations this is equivalent to requiring that α be in some Zariski closed subset of k^{sd^2}. Thus we can interpret the space of A-module structures on k^d as a particular affine variety. We denote this variety by $\mathcal{V}_A(d)$. Notice that the general linear group acts on k^d by a change of basis and so the orbits of this action give points in the algebraic variety belonging to the same isomorphism class of modules.

Proposition 5.6. *Let A and B be algebras over an algebraically closed field k and let A separably divide B. If B has tame representation type then A has finite or tame representation type.*

Proof. We follow the ideas of de la Peña used in [dlP91, theorem 1.3] and [dlP96, theorem 4.2].

[dlP91] de la Peña, *Functors preserving tameness*, Fund. Math. **137** (1991), no. 3, 177–185

[dlP96] de la Peña, *Constructible functors and the notion of tameness*, Comm. Algebra **24** (1996), no. 6, 1939–1955

As in the statement of the proposition, let B be a tame algebra over a field k and let A be an algebra that separably divides B via the modules ${}_AM_B$ and ${}_BN_A$. Seeking a contradiction we assume that A is wild.

As A is wild there is a $k\langle u,v\rangle$-A–bimodule Q that is finitely generated and free as a $k\langle u,v\rangle$-module and such that the functor

$$-\underset{k\langle u,v\rangle}{\otimes} Q\colon \operatorname{mod} k\langle u,v\rangle \longrightarrow \operatorname{mod} A$$

preserves indecomposability and non-isomorphism.

Let d be the rank of Q as a $k\langle u,v\rangle$-module so that, in a similar way to the discussion preceding the proposition, Q may be thought of as an element of

$$\tilde{Q} \in k\langle u,v\rangle^{sd^2}$$

If we consider the 1-dimensional $k\langle u,v\rangle$-modules as elements of k^2 where the coordinates represent the actions of u and v then we get a regular map

$$f\colon k^2 \longrightarrow \mathcal{V}_A(d) \subseteq k^{sd^2}$$

given by evaluating $\tilde{Q}$. The important point here is that if $M_{\lambda,\mu}$ is a 1-dimensional $k\langle u,v\rangle$ module then

$$M_{\lambda,\mu} \underset{k\langle u,v\rangle}{\otimes} Q$$

and

$$f(\lambda,\mu)$$

represent the same module. More discussion on this idea can be found in [DS86].

[DS86] Dowbor and Skowroński, *On the representation type of locally bounded categories*, Tsukuba J. Math. **10** (1986), no. 1, 63–72

Notice that no two points in k^2 represent isomorphic $k\langle u,v\rangle$-modules, hence the map f is injective and $\dim f(k^2) = 2$. We now draw a contradicition to this by showing the dimension must be at most 1.

We first note that by the choice of Q (and the assumption of wildness) we know that $f(k^2)$ contains at most one representative of each isomorphism class of A-modules and that this image can only contain indecomposable modules. Also notice that there exists an integer c such that if X is an A-module of dimension d then $X \underset{A}{\otimes} M$ has dimension less than cd.

Let X be a d-dimensional indecomposable A-module. We have that X is isomorphic to a summand of $X \underset{A}{\otimes} M \underset{B}{\otimes} N$ and hence there is an indecomposable summand Y of $X \underset{A}{\otimes} M$ such that X is isomorphic to a summand of $Y \underset{B}{\otimes} N$.

The dimension of $X \underset{A}{\otimes} M$ is less than cd and for each $d' < cd$ there is a finite set of $k[t]$-B–bimodules

$$\{Y_1^{d'}, Y_2^{d'}, \ldots, Y_{r_{d'}}^{d'}\}$$

such that any indecomposable B-module of dimension d' is isomorphic to

$$S_\lambda \underset{k[t]}{\otimes} Y_i^{d'}$$

for some simple $k[t]$-module

$$S_\lambda \cong \frac{k[t]}{(t-\lambda)}$$

and some $1 \leq i \leq r_{d'}$.

This means that there is a finite set of $k[t]$-A–bimodules (indexed by a finite set I)

$$\{X_i\}_{i \in I} = \{Y_i^{d'} \underset{B}{\otimes} N \mid d' < cd, 1 \leq i \leq r_{d'}\}$$

with the property that X is isomorphic to a summand of

$$S_\lambda \underset{k[t]}{\otimes} X_i$$

for some $\lambda \in k$ and some $i \in I$.

For each $i \in I$ define

$$C_i = \left\{ \big(\lambda, X\big) \,\middle|\, X \text{ is a summand of } S_\lambda \underset{k[t]}{\otimes} X_i \right\} \subseteq k \times f(k^2)$$

and consider the projections

$$\begin{array}{ccc} C_i & \xrightarrow{\pi_1^i} & k \\ {\scriptstyle \pi_2^i}\downarrow & & \\ f(k^2) & & \end{array}$$

Since $f(k^2)$ contains at most one representative of each isomorphism class the Krull–Schmidt theorem tells us that the preimage of π_1^i is finite for each $\lambda \in k$. Hence

$$\dim C_i \leq \dim k = 1$$

Considering the other projections: we have that

$$\bigcup_{i \in I} \pi_2^i(C_i) = f(k^2)$$

is a finite union and so $\dim f(k^2) \leq 1$ giving a contradiction and implying that A cannot be wild. □

As a direct corollary of propositions 5.5 and 5.6 we have the following theorem.

Theorem 5.7:

> If A and B are separably equivalent algebras over an algebraically closed field k then A and B have the same representation type: finite, tame or wild.

5.5 Domestic and Polynomial Growth

Algebras of tame representation type can be broken down further into subcategories. These subdivisions are classified by how many families of modules are required to cover the indecomposables of an algebra. As such we begin with some notation to represent how many families of modules are required in each dimension.

Let A be a tame algebra. Definition 5.2 of tameness indicates that for each dimension $d \in \mathbb{N}$ we have a collection of $k[t]$-A–bimodules satisfying certain properties. We denote by

$$\mu_A(d) \in \mathbb{N}$$

the minimum number of these modules required to satisfy the definition.

Definition 5.8 (Domestic algebra)**.** An algebra A is said to be *domestic* if there is some integer N such that $\mu_A(n) \leq N$ for all positive integers n.

Definition 5.9 (Polynomial growth)**.** An algebra A is said to be of *polynomial growth* if there are some positive integers C and γ_A such that

$$\mu_A(n) \leq C n^{\gamma_A}$$

for all positive integers n. If the integer γ_A is chosen minimally with respect to the definition then it is called the *growth rate*.

There is another characterisation of tameness of an algebra using the theory of *generic modules* and this idea will allow us to prove that separable equivalence preserves not only the broad categories of finite, tame and wild but also these subdivisions of tame.

Recall that if M_A is a right A-module then M is naturally a left module for its endomorphism ring $\mathrm{End}(M_A)$ (in fact it is an $\mathrm{End}(M_A)$-A–bimodule).

Definition 5.10 (Endolength)**.** Let M_A be an A-module. We say that the *endolength* of M is its length when considered as a module for its endomorphism ring and denote this by $\operatorname{end\,len}(M)$. We say that the module is *endofinite* if it has finite endolength.

Definition 5.11 (Generic module)**.** An indecomposable A-module M is said to be a *generic module* if it has infinite length over A but has finite length over $\mathrm{End}(M_A)$.

Generic modules are intricately linked to the 1-parameter families of modules we have seen in the definition of tameness as the next theorem can attest.

[CB91] Crawley-Boevey, *Tame algebras and generic modules*, Proc. London Math. Soc. (3) **63** (1991), no. 2, 241–265

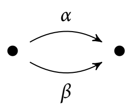
Theorem 5.12: *Crawley-Boevey,* [CB91, 5.7]

For an algebra A, let $g_A(n)$ denote the number of isomorphism classes of generic A-modules of endolength n. Then

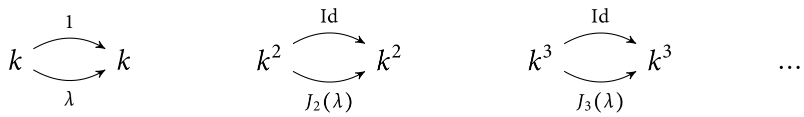
$$\mu_A(n) = \sum_{d|n} g_A(d)$$

The full interaction between 1-parameter families and generic modules is highly complex and so rather than delve fully into the theory we aim to gain an understanding via an example.

Example. Returning to the example of a tame algebra we saw on page 51: let A be the path algebra for the quiver



Recall that A has a single 1-parameter family in each even dimension



and thus theorem 5.12 tells us that we should have exactly one generic module and its endolength is 2.

We claim that the module G,

$$G = k(t) \underset{t}{\overset{1}{\rightrightarrows}} k(t)$$

is a generic module, where $k(t)$ is the field of rational functions.

First let us consider an endomorphism of this module. This is given by a pair of k-endomorphisms f and g, such that the squares $1f = g1$ and $tf = gt$ commute.

$$\begin{array}{ccc} k(t) & \underset{t}{\overset{1}{\rightrightarrows}} & k(t) \\ \downarrow f & & \downarrow g \\ k(t) & \underset{t}{\overset{1}{\rightrightarrows}} & k(t) \end{array}$$

Commutativity with 1 simply means that $f = g$. We will show that commutativity with t means that f is determined by $f(1)$ and hence $\mathrm{End}(G_A) \cong k(t)$.

If $f(1) = r \in k(t)$ then since f commutes with t

$$f(t^n) = t^n f(1) = t^n r$$

and thus if $p \in k[t]$ is a polynomial then

$$f(p) = pr.$$

Similarly by the commutativity with polynomials

$$r = f(1) = f(p/p) = pf(1/p)$$

and so

$$f(1/p) = r/p.$$

So in particular $\dim_{k(t)}(G) = 2$ and G is endofinite.

Now we need only show that G is indecomposable. Seeking a contradiction let us assume that G splits. Using the identity arrow we can identify the two copies of

$k(t)$ and so we must have that $k(t)$ splits as a $k[t]$-module. This means there is a non-trivial idempotent in $\mathrm{End}_{k[t]}\, k(t) \cong \mathrm{End}_{k(t)}\, k(t) \cong k(t)$, but since $k(t)$ is a field this cannot be the case.

Given this generic module it is clear to see how we obtain the family of 2-dimensional modules. In general if G is a generic module then we may find a $k[t]$-A–bimodule M such that

$$G \cong k(t) \underset{k[t]}{\otimes} M$$

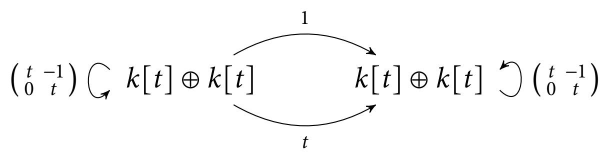
then G provides us with the further families of modules given by $k(t) \underset{k[t]}{\otimes} {}^nM$ where ${}^nM$ is the module given by

$${}^nM = \overbrace{M \oplus M \oplus \cdots \oplus M}^{n\text{-copies}}$$

with the action

$$(m_1, m_2, \ldots, m_n)t = (m_1 t, m_2 t - m_1, \ldots, m_n t - m_{n-1})$$

In this example the 4-dimensional family would be provided by



with the loops representing the action of t. Since the action here is via Jordan matrices, localising this module at each $(t - \lambda)$ for $\lambda \in k$ gives the modules of the 1-parameter family seen earlier.

As immediate corollaries to theorem 5.12 we can provide alternative definitions of tame, domestic and polynomial growth, in terms of the number of generic modules.

Corollary 5.13. *An algebra A is tame if and only if $g_A(n) < \infty$ for all $n \in \mathbb{N}$.*

Corollary 5.14. *An algebra A is of polynomial growth if and only if there are integers C and δ such that*

$$g_A(n) \leq Cn^{\delta}$$

for all positive integers n.

Corollary 5.15. *An algebra A is domestic if and only if it has only finitely many generic modules.*

In order to prove that separable equivalence preserves these finer-grain classes of representation type we will need the following theorem on decomposability of endofinite modules.

Theorem 5.16: *Endofinite Decomposability*

If M is an endofinite module then there is a finite set of indecomposable modules M_i, and cardinals κ_i, such that M decomposes as

$$M \cong M_1^{(\kappa_1)} \oplus \cdots \oplus M_n^{(\kappa_n)}.$$

Moreover if $M_i \ncong M_j$ for all $i \neq j$ then $\operatorname{end len}(M) = \sum_{i=1}^{n} \operatorname{end len}(M_i)$.

The notation $M^{(\kappa)}$ denotes the direct sum of κ copies of M.

For details on the proof of this theorem see [Pre09, 4.4.29], which is also a very good reference on endofinite modules in general.

[Pre09] Prest, *Purity, spectra and localisation*, Encyclopedia of Mathematics and its Applications, vol. 121, Cambridge University Press, Cambridge, 2009

We require one final lemma before we state and prove the theorem.

Lemma 5.17. *Let ${}_AM_B$ be a finitely generated bimodule. There is a constant c_M such that if X_A is endofinite then $X \underset{A}{\otimes} M$ has endolength*

$$\operatorname{end len}(X \underset{A}{\otimes} M) \leq c_M \operatorname{end len}(X).$$

Proof. As M is finitely generated there is an integer n and a left A-epimorphism $A^n \to M$. This map gives a sequence

$$\begin{array}{ccccc} X \underset{A}{\otimes} A^n & \longrightarrow & X \underset{A}{\otimes} M & \longrightarrow & 0 \\ \wr\| & & & & \\ X^n & & & & \end{array}$$

that is exact when considered as left modules for $\operatorname{End}_A(X)$. Thus $n \operatorname{end len}(X)$ bounds the length of $X \underset{A}{\otimes} M$ as an $\operatorname{End}_A(X)$-module.

If we have a chain of $\operatorname{End}_B(X \underset{A}{\otimes} M)$ modules

$$0 = M_r \lneq \cdots \lneq M_1 \lneq X \underset{A}{\otimes} M$$

then this can be considered as a chain of $\operatorname{End}_A(X)$ modules via the canonical homomorphism

$$\begin{array}{ccc} \operatorname{End}_A(X) & \longrightarrow & \operatorname{End}_B(X \underset{A}{\otimes} M) \\ \phi & \mapsto & \phi \underset{A}{\otimes} M \end{array}$$

and hence $\operatorname{end\,len}(X \underset{A}{\otimes} M) \leq n \operatorname{end\,len}(X)$. □

Theorem 5.18:

Let A and B be finite dimensional algebras over an algebraically closed field k such that A separably divides B.

(a) If B is of polynomial growth then A is of polynomial growth.

(b) If B is domestic then A is domestic.

In particular the properties of domestic and polynomial growth are preserved under separable equivalence.

Proof. We will first prove part (a):

Let ${}_AM_B$ and ${}_BN_A$ be the modules providing the separable division.

Denote the generic B-modules of endolength d by

$${}^dG_1, {}^dG_2, \dots, {}^dG_{g_B(d)}.$$

If H is a generic A-module of endolength d then by theorem 5.16 and lemma 5.17

$$H \underset{A}{\otimes} M \cong \bigoplus_{j=1}^{m} {}^{d_j}G_{i_j}^{(\kappa_j)} \oplus F$$

with $1 \leq m \leq c_M d$ and $d_j \leq c_M d$ for all j and for some finite length module F. That this decomposition is essentially unique follows from section 4 of [CB92], in particular see the remarks following proposition 4.5.

[CB92] Crawley-Boevey, *Modules of finite length over their endomorphism rings*, Representations of algebras and related topics (Kyoto, 1990), London Math. Soc. Lecture Note Ser., vol. 168, Cambridge Univ. Press, Cambridge, 1992, pp. 127–184

We have that H is a summand of $H \underset{A}{\otimes} M \underset{B}{\otimes} N$. If H is a summand of $F \underset{B}{\otimes} N$ then H has finite length, which is a contradiction as H is generic. Therefore H is a summand of ${}^{d_j}G_{i_j} \underset{B}{\otimes} N$ for some j.

Now define $\mathcal{H}(d)$ as follows:

$$\mathcal{H}(d) = \left\{ H \in \operatorname{mod} A \;\middle|\; \begin{array}{l} H \text{ a generic summand of } {}^{d'}G_i \underset{B}{\otimes} N \\ d' \leq c_M d,\ 1 \leq i \leq g_B(d') \end{array} \right\}.$$

Thus if H is any generic module with $\operatorname{end\,len}(H) \leq d$ then $H \in \mathcal{H}(d)$.

It is now enough to bound the cardinality of $\mathcal{H}(d)$ by a polynomial in d.

The number of distinct endofinite summands of ${}^{d'}G_i \underset{B}{\otimes} N$ is bounded above by its endolength and hence by $c_N d'$. The number of generic modules of endolength d' is given by $g_B(d')$. We have

$$\begin{aligned} |\mathcal{H}(d)| &\leq \sum_{d' \leq c_M d} (c_N d') g_B(d') \\ &\leq \sum_{d' \leq c_M d} (c_N d') C d'^{\delta} && \text{as } B \text{ is of polynomial growth} \\ &\leq (c_M d)(c_N c_M d) C (c_M d)^{\delta} \\ &\leq C' d^{\delta+2} \end{aligned}$$

and hence A is of polynomial growth.

For part (b) we may start by following the same proof through to the definition of $\mathcal{H}$.

As B is domestic there are only finitely many generic modules, thus there is some integer d for which $g_B(d') = 0$ for all $d' > d$. In particular

$$\mathcal{H}(d) = \mathcal{H}(d')$$

for all $d' > d$. Thus every generic A module is in the finite set $\mathcal{H}(d)$. □

5.6 Examples of inequivalence

The discussion at the end of section 4.1 led us to ask whether or not there exists cyclic groups C_{p^n} and C_{p^m} such that their group algebras over a field of characteristic p are separably equivalent. Unfortunately the best we can offer is a partial solution to this problem. Specifically we will demonstrate the inequivalence of the group algebras for several small cyclic groups, leaving the general question wide open.

As we have seen previously (over a field of characteristic p) there is an isomorphism between the group algebra of a cyclic p-group and a truncated polynomial algebra. If we let

$$\Lambda_n = \frac{k[x]}{(x^n)}$$

denote the truncated polynomial algebra of length n then over a field of characteristic p we have an isomorphism $kC_{p^n} \cong \Lambda_{p^n}$. By phrasing the above question in terms of Λ_n we can achieve results for a general field that will then give answers for the cyclic group case over fields with the correct characteristic.

The algebra Λ_n has finite representation type with exactly n indecomposable modules $M_i \cong x^{n-i}\Lambda_n$:

$$\begin{array}{cccc} M_1 & M_2 & & M_n \\ k & k & & \left.\begin{array}{c} k \\ \vdots \\ k \end{array}\right\} n\text{-times} \\ & k & \cdots & \end{array}$$

Let us construct the Auslander–Reiten quiver for this algebra (see section 2.5), so for each i and j we wish to calculate a basis for

$$\frac{\mathrm{rad}(M_i, M_j)}{\mathrm{rad}^2(M_i, M_j)}$$

If $i = j$ then $\mathrm{rad}(M_i, M_i)$ is generated by the map given by multiplication by x but notice that this map is the same as the composition

$$\begin{array}{ccccc} M_i & & M_{i-1} & & M_i \\ k & \longrightarrow & k & \searrow & k \\ \vdots & & \vdots & & k \\ k & \longrightarrow & k & \searrow & \vdots \\ k & & & & k \end{array}$$

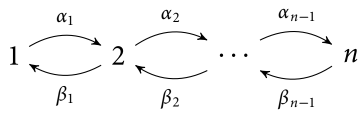
which is in $\mathrm{rad}^2(M_i, M_i)$ and hence we have no self-arrows in the quiver. Next consider the case $j = i - 2$, the surjective homomorphism factors through M_{i-1}:

$$\begin{array}{ccccc} M_i & & M_{i-1} & & M_{i-2} \\ k & \longrightarrow & k & \longrightarrow & k \\ \vdots & & \vdots & & \vdots \\ k & \longrightarrow & k & \longrightarrow & k \\ k & \longrightarrow & k & & \\ k & & & & \end{array}$$

and so again we have no arrows in the Auslander–Reiten quiver between M_i and M_{i-2}. Similar treatment shows that we have no arrows $M_i \to M_{i+2}$. We are left to consider maps $M_i \to M_{i\pm1}$. The surjection $\beta_{i-1}\colon M_i \to M_{i-1}$ and the inclusion $\alpha_i\colon M_i \to M_{i+1}$ factor through no other indecomposable module and hence these maps give arrows in the Auslander–Reiten quiver. Notice however that any other map factors through a map in $\operatorname{rad}(M_i, M_i)$ and hence does not contribute an arrow.

Let us now consider the relations these maps satisfy. The path $\beta_{i-1}\alpha_{i-1}$ represents the composition we saw above (which corresponds to multiplication by x). The path $\alpha_i\beta_i$ represents this same map and hence we have the relations $\alpha_i\beta_i = \beta_{i-1}\alpha_{i-1}$ for $1 < i < n$ and $\alpha_1\beta_1 = 0$.

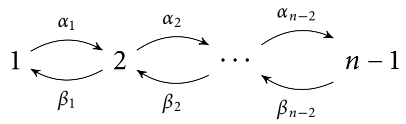
We have calculated that the Auslander–Reiten quiver for Λ_n is



$$\alpha_1\beta_1 = 0$$
$$\alpha_i\beta_i = \beta_{i-1}\alpha_{i-1} \text{ for } 1 < i < n$$

If kQ_n is the Auslander algebra of Λ_n then the category $\operatorname{mod} kQ_n$ is equivalent to the category $\operatorname{Fun}(\operatorname{mod}\Lambda_n, \operatorname{mod} k)$. This fact together with proposition 4.20 means that if Λ_n and Λ_m are separably equivalent then kQ_n and kQ_m are separably equivalent.

If we instead restrict to the stable category, we must add additional relations for all maps that factor through a projective module. In this example we have a single projective module Λ_n thus we must add the relation $\beta_{n-2}\alpha_{n-2} = 0$ and remove the vertex n.



$$\alpha_1\beta_1 = \beta_{n-2}\alpha_{n-2} = 0$$
$$\alpha_i\beta_i = \beta_{i-1}\alpha_{i-1} \text{ for } 1 < i < n-1$$

The path algebra for the quiver with relations given above is called the preprojective algebra of type A_{n-1}. Here the A_{n-1} refers to the Dynkin diagram and note that preprojective algebras can be defined for many different quivers (see section 3 of [GLS05] for more details). Now proposition 4.21 tells us that if Λ_n and Λ_m are separably equivalent then the preprojective algebras of type A_{n-1} and A_{m-1} are separably equivalent.

[GLS05] Geiss, Leclerc, and Schröer, *Semicanonical bases and preprojective algebras*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 2, 193–253

Theorem 5.19:

> The algebras
>
> $$\Lambda_n = \frac{k[x]}{(x^n)} \qquad \Lambda_m = \frac{k[x]}{(x^m)}$$
>
> are not separably equivalent for positive integers $n \leq 6$ and $m \neq n$.

Proof. We prove the claim by handling each n on a case-by-case basis.

$n = 1$

The algebra Λ_1 is isomorphic to the field k and so if Λ_m is separably equivalent to Λ_1 then according to proposition 4.2 Λ_m must be separable and hence semisimple. It is clear that for any $m > 1$ the algebra Λ_m is not semisimple and hence the algebras are separably inequivalent.

$n = 2$*

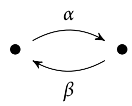
*Note that Linckelmann gives an alternative proof of the $n = 2$ case in example 10.7 of [Lin11a].

[Lin11a] Linckelmann, *Cohomology of block algebras of finite groups*, Representations of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, pp. 189–250

If Λ_2 is separably equivalent to Λ_m for some $m > 2$ then by the discussion above we see that this gives a separable equivalence of the preprojective algebras of types A_1 and A_{m-1}. The first of these is isomorphic to the base field k, but it is clear that for $m > 2$ the preprojective algebra A_{m-1} is not semisimple, therefore the algebras cannot be separably equivalent.

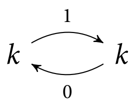
$n = 3$

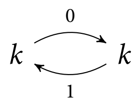
We denote by Γ_n the preprojective algebra of type A_n. As in the last two cases Λ_3 and Λ_m being separably equivalent means that Γ_2 and Γ_{m-1} are separably equivalent. From proposition 4.21 we see that this means the categories $\mathrm{Fun}(\underline{\mathrm{mod}}\,\Gamma_2, \mathrm{mod}\,k)$ and $\mathrm{Fun}(\underline{\mathrm{mod}}\,\Gamma_{m-1}, \mathrm{mod}\,k)$ are separably equivalent. Now Γ_2 is the path algebra of



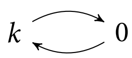
$$\alpha\beta = \beta\alpha = 0$$

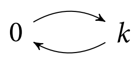
with the projective modules given by





Thus in $\underline{\mathrm{mod}}\,\Gamma_2$, once we have factored out projective modules, we are left with just the simple modules





with no non-trivial maps between them. This shows that $\mathrm{Fun}(\underline{\mathrm{mod}}\,\Gamma_2, \mathrm{mod}\,k)$ is equivalent to the representations of the quiver with two vertices and no arrows: a semisimple algebra.

To show that $\mathrm{Fun}(\underline{\mathrm{mod}}\,\Gamma_{m-1}, \mathrm{mod}\,k)$ is not semisimple when $m > 2$ we need only demonstrate that the AR-quiver of Γ_{m-1} contains an arrow between two vertices representing non-projective modules. We refer to the work of Geiss, Leclerc and Schröer in [GLS05]. Proposition 3.3 of this text tells us that Γ_m has infinitely many isomorphism classes of indecomposable modules when $m > 4$, thus the AR-quiver must contain an arrow between two non-projectives. For smaller m, section 20 gives the explicit AR-quivers and shows that the AR-quivers of both Γ_3 and Γ_4 contain an arrow between non-projectives.

[GLS05] Geiss, Leclerc, and Schröer, *Semicanonical bases and preprojective algebras*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 2, 193–253

$n = 4$

If Λ_4 and Λ_m were separably equivalent then by proposition 4.6 we would have that their tensor products with a fixed third algebra would be equivalent also. For an algebra A, the algebra of upper triangular matrices with entries from A

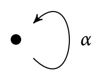
$$\begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$$

is called the 2×2 triangular matrix algebra, $T_2(A)$. This is isomorphic to the tensor product of A with the path algebra $1 \xrightarrow{\alpha} 2$ via the isomorphism

$$\begin{array}{rcl} A \otimes \left(1 \xrightarrow{\alpha} 2\right) & \longrightarrow & T_2(A) \\ a \otimes e_1 & \mapsto & \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) \\ a \otimes e_2 & \mapsto & \left(\begin{smallmatrix} 0 & 0 \\ 0 & a \end{smallmatrix}\right) \\ a \otimes \alpha & \mapsto & \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right) \end{array}$$

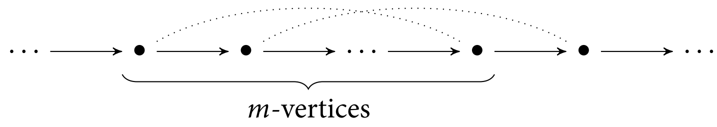
The representation types of algebras of this form were classified in [LS00]. These were classified via lists of quivers that may appear as a factor algebra of a subquiver of a Galois covering. The relevant sections are theorems 1 and 4, together with the lists of quivers in sections 2 and 5. The algebra Λ_m is the path algebra for the quiver

[LS00] Leszczyński and Skowroński, *Tame triangular matrix algebras*, Colloq. Math. **86** (2000), no. 2, 259–303



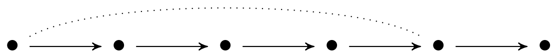
$$\alpha^m = 0$$

The Galois covering of this is given by the quiver



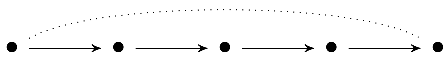
[Gab81] Gabriel, *The universal cover of a representation-finite algebra*, Representations of algebras (Puebla, 1980), Lecture Notes in Math., vol. 903, Springer, Berlin-New York, 1981, pp. 68–105

with dotted lines representing the relation that the path is zero. For more details on this example and Galois coverings in general see [Gab81, 2.8ff.]. The key point to note regarding this quiver is that for $m > 4$ the Galois covering contains a subquiver with



[LS00] Leszczyński and Skowroński, *Tame triangular matrix algebras*, Colloq. Math. **86** (2000), no. 2, 259–303

as a factor algebra, which is [LS00, 2.74] and this means that $T_2(\Lambda_m)$ is wild. When $m = 4$ there is no subquiver containing a factor algebra of wild type but it does contain



as a factor algebra, which is [LS00, 5.12] and shows that $T_2(\Lambda_4)$ is tame. Finally we can check that no quiver in section 5 of the text appears as a subfactor of the Galois quiver when $m < 4$ and hence for these $T_2(\Lambda_m)$ are of finite type. We have

$T_2(\Lambda_n)$ has	finite representation type for	$n < 4$
	tame	$n = 4$
	wild	$n > 4$

Since $T_2(\Lambda_4)$ and $T_2(\Lambda_m)$ have different representation type for $m > 4$ theorem 5.7 tells us that Λ_4 and Λ_m cannot be separably equivalent. Note that this method also gives an alternative proof for the case $n = 3$.

$n = 5$ and $n = 6$

[GLS05] Geiss, Leclerc, and Schröer, *Semicanonical bases and preprojective algebras*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 2, 193–253

Finally we can see from [GLS05, proposition 3.3] that

Γ_n has	finite representation type for	$n < 5$
	tame	$n = 5$
	wild	$n > 5$

and since a separable equivalence of Λ_n and Λ_m induces an equivalence of Γ_{n-1} and Γ_{m-1} we see that when $n \in \{5, 6\}$ and $m \neq n$ then Λ_n and Λ_m cannot be separably equivalent. □

Theorem 5.19 is a long way from answering the question as to whether or not algebras for cyclic groups can be separably equivalent. It does however demonstrate many examples of how one can show that algebras are not separably equivalent, using many of the propositions of the preceding sections. The proof of the theorem uses representation type to differentiate between algebras for $n \leq 6$. For larger n all the algebras we have constructed from Γ_n have wild representation type and so it would appear new methods will be required to show the inequivalence of these algebras. We conclude with the corollaries:

Corollary 5.20. *Let k be a field of characteristic 2. The group algebras kC_2, kC_4 and kC_{2^n} are pairwise separably inequivalent for any $n > 2$.*

Corollary 5.21. *Let k be a field of characteristic 3. The group algebras kC_3 and kC_{3^n} are separably inequivalent for any $n > 1$.*

Corollary 5.22. *Let k be a field of characteristic 5. The group algebras kC_5 and kC_{5^n} are separably inequivalent for any $n > 1$.*

6 Representation dimension

The representation dimension of an algebra measures how far the algebra is from being of finite representation type. Auslander introduced the concept in [Aus71] and proved that an algebra is semisimple if and only if its representation dimension is equal to 1, and otherwise an algebra is of finite representation type if and only if its representation dimension is 2.

[Aus71] Auslander, *Representation dimension of Artin algebras*, Queen Mary College Mathematics Notes (1971), 70,

The purpose of this chapter is to use the separable equivalence between a group algebra and the group algebra of a Sylow p-subgroup (proposition 4.5) to give an upper bound for the representation dimension of group algebras for groups with elementary abelian Sylow subgroups. We will present everything in this chapter in terms of groups and their Sylow p-subgroups however all the results generalise easily to the case of blocks and defect groups.

Throughout this chapter we will only deal with finite dimensional algebras over algebraically closed fields and finitely generated modules. We begin with several definitions that will lead on to the definition of representation type.

Definition 6.1 (Projective dimension)**.** Let M be an A-module and let

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a projective resolution of M. We say that the resolution has *length* n if $P_n \neq 0$ but $P_i = 0$ for all $i > n$. If this property does not hold for any n, then the resolution is of *infinite length*.

The *projective dimension* $\mathrm{pd}(M)$, is defined to be the minimal length of a projective resolution.

Definition 6.2 (Global dimension)**.** Let A be a k-algebra. The *global dimension* of A, denoted by $\mathrm{gl\,dim}(A)$, is defined to be the supremum of the projective dimensions of all A-modules.

$$\operatorname{gl}\dim(A) = \sup\{\operatorname{pd}(M) \mid M \text{ an } A\text{-module}\}$$

Definition 6.3 (Generator/cogenerator)**.** Let A be a k-algebra. A module M is said to *generate* the module category mod A, if for any module N there is an integer n and an epimorphism

$$M^n \longrightarrow N \longrightarrow 0.$$

A module M is said to *cogenerate* the module category if for any module N there is an integer n and a monomorphism

$$0 \longrightarrow N \longrightarrow M^n.$$

Notice that M is a generator if and only if M contains each indecomposable projective module as a direct summand. Similarly, M is a cogenerator if it contains each indecomposable injective module as a direct factor. In the case of self-injective algebras, such as group algebras, these two properties are equivalent.

Definition 6.4 (Representation dimension)**.** Let A be a k-algebra. If A is semisimple then the *representation dimension* $\operatorname{rep}\dim(A)$, is defined to be 1, otherwise the representation dimension of A is defined as follows:

$$\operatorname{rep}\dim(A) = \inf\left\{\operatorname{gl}\dim\left(\operatorname{End}_A(M)\right) \mid M \text{ generates and cogenerates mod } A\right\}$$

It is an open question as to whether or not representation dimension is preserved under separable equivalence, however it is known to be preserved for representation dimension of 1 or 2. This fact is a direct consequence of the following result of Auslander ([Aus71]) together with proposition 5.5.

[Aus71] Auslander, *Representation dimension of Artin algebras*, Queen Mary College Mathematics Notes (1971), 70,

Proposition 6.5. *Let A be an algebra over an algebraically closed field k.*

(a) $\operatorname{rep}\dim(A) = 1$ *if and only if A is semisimple;*

(b) $\operatorname{rep}\dim(A) \leq 2$ *if and only if A is of finite representation type.*

[Iya03] Iyama, *Finiteness of representation dimension*, Proc. Amer. Math. Soc. **131** (2003), no. 4, 1011–1014 (electronic)

[Rou06] Rouquier, *Representation dimension of exterior algebras*, Invent. Math. **165** (2006), no. 2, 357–367

Auslander also showed in [Aus71] that the Loewy length of a selfinjective algebra provides an upper bound for its representation dimension. Iyama showed in [Iya03] that the representation dimension of a general algebra is always finite, whilst in [Rou06] Rouquier simultaneously gave the first example of an algebra with representation dimension greater than 3 and showed that representation dimension can be arbitrarily large. Work of Bergh and Erdmann in [BE11] provide better bounds in the case of Hecke algebras and group algebras of symmetric groups using the idea of separable equivalence and passing to a Sylow p-subgroup. In order to use similar techniques we will first define a set of modules for an elementary abelian p-group. The direct sum of the modules in this set will provide a generator of the module category and it is this module we will use to provide an upper bound for the representation dimension.

[BE11] Bergh and Erdmann, *The representation dimension of Hecke algebras and symmetric groups*, Adv. Math. **228** (2011), no. 4, 2503–2521

6.1 Bounding representation dimension

We will aim to bound the representation dimension of group algebras. We saw in proposition 4.5 that over a field of characteristic p a group algebra is separably equivalent to the group algebra of a Sylow p-subgroup. If P is a p-group we may use this separable equivalence together with the following theorem of Bergh and Erdmann to find a generator of kP whose global dimension will simultaneously bound the representation dimension of all group algebras kG where P is a Sylow p-subgroup of G.

Theorem 6.6: [BE11, theorem 2.3]

Let A and B be finite dimensional algebras and suppose there exists a B-module M such that

(a) A separably divides B through ${}_AX_B$ and ${}_BY_A$; and

(b) $\mathrm{Hom}_A(Y, M \underset{B}{\otimes} Y) \in \mathrm{add}_B\, M$

then $\mathrm{gl\,dim}\,\mathrm{End}_A(M \underset{B}{\otimes} Y) \leq \mathrm{gl\,dim}\,\mathrm{End}_B(M)$.

If we have that P is a Sylow p-subgroup of G then the separable equivalence of kP and kG is through the bimodules ${}_{kG}kG_{kP}$ and ${}_{kP}kG_{kG}$. In this context theorem 6.6 becomes the following corollary.

Corollary 6.7. *Let P be a Sylow p-subgroup of G and k a field of characteristic p. If M is a kP-module such that $M{\uparrow}^G{\downarrow}_P \in \mathrm{add}\, M$ then*

$$\mathrm{gl\,dim}\,\mathrm{End}_{kG}(M{\uparrow}^G) \leq \mathrm{gl\,dim}\,\mathrm{End}_{kP}(M).$$

If we can find a generator M of kP such that add M is closed under induction to any supergroup G and restriction back down to P then since $M\uparrow^G$ is a generator of kG the representation dimension of kG is bounded above by the global dimension of $\mathrm{End}_{kP}(M)$.

Let us assume that for a p-group P we have a finite set of modules $\mathcal{M}_P$ with the properties:

(res-ind): if $X \in \mathcal{M}_P$ and L is a subgroup of P then $X\downarrow_L\uparrow^P \in \mathcal{M}_P$;

(isom): if H and L are subgroups of P and there is an isomorphism $\phi\colon H \xrightarrow{\sim} L$ then

$$\phi(\mathcal{M}_P\downarrow_L) = \mathcal{M}_P\downarrow_H,$$

where $\phi(\mathcal{M}_P\downarrow_L)$ denotes the set of H-modules obtained via ϕ by restriction of scalars.

For any supergroup G of P, Mackey decomposition gives us

$$M\uparrow^G\downarrow_P \cong \bigoplus_{s\in P\backslash G/P} (M\otimes s)\downarrow_{s^{-1}Ps\cap P}\uparrow^P$$

and so if

$$M = \bigoplus_{X\in\mathcal{M}_P} X$$

then the properties (res-ind) and (isom) mean that add M is closed under induction-restriction. We can therefore use M to simultaneously bound the representation dimension of all group algebras for groups with P as a Sylow p-subgroup.

6.2 Elementary abelian groups

In this section we will define a class of modules for elementary abelian groups that is closed under induction-restriction and that contains a generator of the group algebra (the regular module). Using the remarks made at the end of the last section we will use this class of modules to bound the representation dimension for all group algebras with the given elementary abelian group as a Sylow p-subgroup.

Definition 6.8 ($\mathcal{M}_P$)**.** Let P be an elementary abelian p-group and k a field of characteristic p. Let $\mathcal{M}_P$ be the set of indecomposable kP-modules that is minimal with respect to the following properties:

(a) $kP \in \mathcal{M}_P$;

(b) if $X \in \mathcal{M}_P$, H is a subgroup of P and Z is an indecomposable summand of $X{\downarrow_H}{\uparrow^P}$ then $Z \in \mathcal{M}_P$;

(c) if $X \in \mathcal{M}_P$, d is a positive integer and Z is an indecomposable summand of $X/\operatorname{rad}^d X$ then $Z \in \mathcal{M}_P$.

Remark. Notice that by definition $\mathcal{M}_P$ satisfies the (res-ind) property of the last section. It is also clear from the symmetry in the definition that $\mathcal{M}_P$ is closed under automorphisms of P. Given any pair of isomorphic subgroups of P there is an automorphism of P mapping one to the other and we therefore also have that P is closed under the (isom) property. Thus if $\mathcal{M}_P$ is finite (which we show in proposition 6.16) and we let

$$M = \bigoplus_{X \in \mathcal{M}_P} X$$

then we can calculate an upper bound on the global dimension of $\operatorname{End}(M)$ and this will bound the representation dimension of kG for any group G with Sylow p-subgroup isomorphic to P.

For convenience we will denote the quotient by a power of the radical

$$X_{(m)} = \frac{X}{\operatorname{rad}^m X}.$$

Additionally we define the following classes:

$$\mathcal{M}_{P\downarrow H} = \left\{ Z \;\middle|\; Z \text{ an indecomposable summand of } X{\downarrow_H} \text{ for } X \in \mathcal{M}_P \right\}$$
$$\mathcal{M}_{H\uparrow P} = \left\{ Z \;\middle|\; Z \text{ an indecomposable summand of } X{\uparrow^P} \text{ for } X \in \mathcal{M}_H \right\}$$

We will continue by proving several useful properties of this class of modules. To do this it will be convenient to define an integer-valued property ℓ for each indecomposable module in $\mathcal{M}_P$. We construct modules beginning with kP and applying a finite number of steps $X \mapsto Z$ where

(a) Z is an indecomposable summand of $X{\downarrow_H}{\uparrow^P}$ for some subgroup $H < P$; or

(b) Z is an indecomposable summand of $X_{(d)}$ for some positive integer d.

It is clear that any module X of $\mathcal{M}_P$ is obtained in this manner and we define $\ell(X)$ to be the smallest number of steps required for this to be true, for example $\ell(kP) = 0$ and $\ell(k) = \ell(kP_{(1)}) = 1$.

Proposition 6.9. *If H is a subgroup of P then $\mathcal{M}_{H\uparrow P} \subseteq \mathcal{M}_P$.*

Proof. Let $X \in \mathcal{M}_H$. We perform induction on $\ell(X)$.

Firstly if $\ell(X) = 0$ then $X = kH$, but $kH\uparrow^P = kP$ and thus the result is true. Now assume that if $Y \in \mathcal{M}_H$ with $\ell(Y) < \ell(X)$ then $Y\uparrow^P \in \mathcal{M}_P$. Note that since we are working inside a p-group, Green's indecomposability theorem (see [Gre59]) means that induction preserves indecomposability. We have that X can be expressed in one of the following two ways: if Y is such that $\ell(Y) = \ell(X) - 1$ then either

[Gre59] Green, *On the indecomposable representations of a finite group*, Math. Z. **70** (1959), 430–445

(a) X is a summand of $Y\downarrow_L\uparrow^H$ for some $L < H$; or

(b) X is a summand of $Y_{(m)}$ for some positive integer m.

(a) We have that Y is a summand of $Y\uparrow^P\downarrow_H$ and so $X\uparrow^P$ is a summand of $Y\uparrow^P\downarrow_L\uparrow^P$ but these summands are in $\mathcal{M}_P$ since $Y\uparrow^P$ is in $\mathcal{M}_P$.

(b) Without loss of generality we can assume that $|P : H| = p$ and that

$$H = \langle g_2, \ldots, g_n \rangle < \langle g_1, g_2, \ldots, g_n \rangle = P$$

If we let $x = (g_1 - 1)$ then the induction of Y to P can be decomposed as H-modules

$$Y\uparrow^P\downarrow_H \cong \bigoplus_{s=0}^{p-1} Y \otimes x^s.$$

Similarly we have

$$\operatorname{rad}^m \left(Y\uparrow^P\right)\downarrow_H \cong \bigoplus_{s=0}^{p-1} \operatorname{rad}^{m-s} Y \otimes x^s$$

where we use the convention $\operatorname{rad}^i Y = Y$ whenever $i \leq 0$. Thus we can put these together and get that

$$Y\uparrow^P_{(m)}\downarrow_H \cong \bigoplus_{s=0}^{\min(p-1,m)} Y_{(m-s)} \otimes x^s$$

In particular this module contains X as a summand. This together with the fact that $Y\uparrow^P$ is in $\mathcal{M}_P$ gives the result. □

Proposition 6.10. *If P is an elementary abelian p-group and H is a subgroup of P then $\mathcal{M}_{P\downarrow H} = \mathcal{M}_H$.*

Proof. For each $Z \in \mathcal{M}_H$, Z is a summand of $Z\uparrow^P\downarrow_H$. This together with proposition 6.9 means that $\mathcal{M}_H \subseteq \mathcal{M}_{P\downarrow H}$. As with the last proposition we prove the reverse inclusion by induction on $\ell(X)$ for $X \in \mathcal{M}_P$. The result is clear when $X = kP$ and so we assume that X is obtained in one step from Y and the result holds for Y.

(a) Let X be a summand of $Y\downarrow_L\uparrow^P$ for some subgroup L of P. Then there is a module $Z \in \mathcal{M}_L$ such that $X \cong Z\uparrow^P$. By Mackey decomposition we then have

$$\begin{aligned} X\downarrow_H &\cong Z\uparrow^P\downarrow_H \\ &\cong \underbrace{Z\downarrow_{L\cap H}\uparrow^H \oplus \cdots \oplus Z\downarrow_{L\cap H}\uparrow^H}_{|P:LH|\text{-copies}} \end{aligned}$$

and summands of this are in $\mathcal{M}_H$ by the induction hypothesis and proposition 6.9.

(b) Let $X \cong Y_{(m)}$ for some positive integer m. We may assume that $Y \cong Z\uparrow^P$ for some $Z \in \mathcal{M}_L$ and a subgroup L of P. Further we may assume without loss of generality that both H and L are index p subgroups of P so that we have

$$\begin{aligned} H &= \langle h, g_3, \ldots, g_n \rangle \\ L &= \langle l, g_3, \ldots, g_n \rangle \end{aligned}$$

Suppose $H \neq L$ so that

$$kP \cong \bigoplus_{i=0}^{p-1} kH \otimes (l-1)^i$$

as H-modules and with the obvious action of l. Similarly we may decompose the quotient

$$kP_{(m)} \cong \bigoplus_{i=0}^{p-1} kH_{(m-i)} \otimes (l-1)^i.$$

We can decompose

$$Y \cong \bigoplus_{j=0}^{p-1} Z \otimes (h-1)^i$$

as L-modules with the obvious h action. Thus

$$\begin{aligned}
X\!\downarrow_H &\cong Y \underset{kP}{\otimes} kP_{(m)}\!\downarrow_H \\
&\cong \left(\bigoplus_{j=0}^{p-1} Z \otimes (h-1)^j\right) \underset{kP}{\otimes} \left(\bigoplus_{i=0}^{p-1} kH_{(m-i)} \otimes (l-1)^i\right)\downarrow_H \\
&\cong \left(\bigoplus_{j=0}^{p-1} Z \otimes (h-1)^j\right) \underset{kP}{\otimes} \left(\bigoplus_{j=0}^{p-1}\bigoplus_{i=0}^{p-1} k[L\cap H]_{(m-i-j)} \otimes (l-1)^i (h-1)^j\right)\downarrow_H \\
&\cong \bigoplus_{j=0}^{p-1}\left(Z \underset{kL}{\otimes} \bigoplus_{i=0}^{p-1}\left(k[L\cap H]_{(m-i-j)} \otimes (l-1)^i\right)\right) \otimes (h-1)^j \\
&\cong \bigoplus_{j=0}^{p-1} Z_{(m-j)} \otimes (h-1)^j
\end{aligned}$$

and so by the induction hypothesis the result holds. In the case that $H = L$ a similar yet much simpler argument applies. □

We will use the module

$$M = \bigoplus_{X \in \mathcal{M}_P} X$$

to bound the representation dimension of kP by calculating an upper bound for the global dimension of $\mathrm{End}_{kP}(M)$. We will go on to show that if G is a group with P as a Sylow p-subgroup then the separable equivalence of kG and kP means that the global dimension of $\mathrm{End}_{kP}(M)$ also bounds the representation dimension of kG.

The bound on the global dimension of $\mathrm{End}_{kP}(M)$ will come as a result of the algebra being *strongly quasi-hereditary* using a result of Ringel, which was in turn based on ideas of Iyama: see [Rin10] and [Iya03].

[Rin10] Ringel, *Iyama's finiteness theorem via strongly quasi-hereditary algebras*, J. Pure Appl. Algebra **214** (2010), no. 9, 1687–1692

[Iya03] Iyama, *Finiteness of representation dimension*, Proc. Amer. Math. Soc. **131** (2003), no. 4, 1011–1014 (electronic)

Definition 6.11 (Strongly quasi-hereditary). Let Γ be a finite dimensional algebra over a field, let $\{S_i\}_{i\in I}$ be the set of simple modules and P_i the projective cover of S_i. We say that Γ is *left strongly quasi-hereditary with n layers* if there is a function called the *layer function*

$$l: I \to \{1, \ldots, n\}$$

such that for each simple module S_i there is an exact sequence

$$0 \to R_i \to P_i \to \Delta_i \to 0$$

satisfying:

(a) $R_i = \bigoplus_{j \in J} P_j$ with $l(j) > l(i)$ for each $j \in J$;

(b) if S_j is a composition factor of $\operatorname{rad} \Delta_i$ then $l(j) < l(i)$.

Theorem 6.12: [Rin10]

[Rin10] Ringel, *Iyama's finiteness theorem via strongly quasi-hereditary algebras*, J. Pure Appl. Algebra **214** (2010), no. 9, 1687–1692

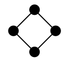
> If Γ is a left-strongly quasi-hereditary algebra with n layers then $\operatorname{gl\,dim}(\Gamma) \leq n$.

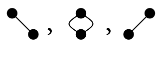
We use the notation $\mathfrak{t}X$ to denote the module $X_{(m)}$ where m is maximal such that $\operatorname{rad}^m X$ is non-zero. If X is semisimple so that $\operatorname{rad} X = 0$ then we say $m = 0$ and $\mathfrak{t}X = 0$. We refer to $m+1$ in this context as the *radical length* of X which we denote by $\operatorname{rad\,len} X$. Note that if $\mathfrak{t}X = X_{(m)}$ then $\mathfrak{t}(\mathfrak{t}X) = X_{(m-1)}$ and in this way any quotient by a power of the radical of X may be written in the form $\mathfrak{t}^i X$ (we say that $\mathfrak{t}^0 X = X$).

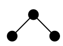
Let P be an elementary abelian p-group. We partition the set $\mathcal{M}_P$ into a sequence of subsets as follows. First we let $\mathcal{M}_P^0 = \{kP\}$. Now assume we have defined $\mathcal{M}_P^i$: we let $\mathcal{M}_P^{i+1}$ be the subset of the remaining modules that are first maximal with respect to radical length, and within these modules we choose those that have minimal dimension.

Example. We highlight this ordering with an example for the group $P = C_2 \times C_2$. We have six modules in $\mathcal{M}_P$

and the classes $\mathcal{M}_P^i$ are then given by

$\mathcal{M}^0 = \{$  $\}$

$\mathcal{M}^1 = \{$  $\}$

$\mathcal{M}^2 = \{$  $\}$

$\mathcal{M}^3 = \{ \bullet \}$

Theorem 6.13:

> Let P be an elementary abelian p-group. If
>
> $$M = \bigoplus_{X \in \mathcal{M}_P} X$$
>
> and $\mathcal{M}_P^n$ is empty then $\operatorname{End}_{kP}(M)^{\mathrm{op}}$ is left strongly quasi-hereditary with at most n layers.

Proof. Let X be a module in $\mathcal{M}_P$ so that

$$P_X = \mathrm{Hom}_{kP}(X, M)$$

is an indecomposable projective $\mathrm{End}_{kP}(M)$-module and let

$$\pi\colon X \to \mathfrak{t}X$$

be the natural projection. Define Δ_X to be the quotient of $\mathrm{Hom}_{kP}(X, M)$ by those maps that factor through π:

$$\Delta_X = \frac{\mathrm{Hom}_{kP}(X, M)}{\{f \circ \pi \mid f\colon \mathfrak{t}X \to M\}}$$

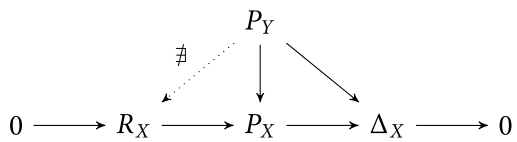
and let $R_X = \mathrm{Hom}(\mathfrak{t}X, M)$. We claim that the short exact sequence

$$0 \longrightarrow R_X \longrightarrow P_X \longrightarrow \Delta_X \longrightarrow 0$$

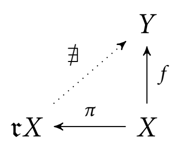
satisfies the properties in the definition of left strongly quasi-hereditary algebras.

(a) That R_X is projective and that if $X \in \mathcal{M}_P^i$ and $\mathfrak{t}X \in \mathcal{M}_P^j$ then $j > i$ is clear.

(b) Assume that the simple module corresponding to $Y \in \mathcal{M}_P^j$ is a composition factor of Δ_X. We have a map $P_Y \to \Delta_X$ that lifts to a map $P_Y \to P_X$ that does not factor through R_X:



By using the correspondence between $\mathrm{add}\, M$ and $\mathrm{End}_{kP}(M)^{\mathrm{op}}$ (see the discussion in section 6.4 but note that here we are using the contravariant version) this gives a map $f\colon X \to Y$ that does not factor through π:



If $j > i$ then either $\mathrm{rad\,len}\, Y < \mathrm{rad\,len}\, X$, or the radical lengths are equal but $\dim Y > \dim X$. In either case if $m+1 = \mathrm{rad\,len}(X)$ then $\mathrm{rad}^m X$ must be in the kernel of f.

Now assume that $j = i$ and f does not factor through π. In this situation the head of X maps onto the head of Y and since the dimensions of X and Y are equal, f must be an isomorphism.

This is enough to show that if Y is a composition factor of rad Δ_X then $j < i$. □

Corollary 6.14. *If $P = C_p \times C_p \times \cdots \times C_p$ is an elementary abelian p-group of order p^n and*

$$M = \bigoplus_{X \in \mathcal{M}_P} X$$

then

$$\operatorname{gl\,dim} \operatorname{End}_{kP}(M) \leq p^n(n(p-1)+1)$$

Proof. First we note that the global dimension of an algebra is equal to the global dimension of its opposite algebra and so we can apply theorem 6.13. The dimension of kP is given by p^n and the radical length of kP is given by $n(p-1)+1$, therefore the number of distinct radical length–dimension pairs is bounded above by $p^n(n(p-1)+1)$. Since the radical length–dimension pairs index the sets $\mathcal{M}_P^i$ this bounds the number of layers and hence the global dimension. □

We are able to define a new collection of modules $\mathcal{N}_P$ and by demonstrating that this is the same as $\mathcal{M}_P$ show that the number of distinct radical length–dimension pairs is much smaller than the bound given in corollary 6.14.

$$\mathcal{N}_P = \left\{ \mathfrak{r}^i\left(X\uparrow^P\right) \,\middle|\, X \in \mathcal{M}_H \text{ with } |P:H| = p \text{ and } 0 \leq i < p \right\}$$

We wish to show that $\mathcal{N}_P$ is the same class of modules as $\mathcal{M}_P$. To prove this we require the following lemma.

Lemma 6.15. *Let P be an elementary abelian p-group and let H be an index p subgroup of P. If X is a kH-module then*

$$\left(X\uparrow^P\right)_{(m)} \cong \left(X_{(m)}\uparrow^P\right)_{(m)}$$

Proof. For a kP-module Y we have that

$$Y_{(m)} \cong Y \underset{kP}{\otimes} kP_{(m)}$$

and so

$$\begin{aligned}\left(X_{(m)}\overset{P}{\uparrow}\right)_{(m)} &\cong X \underset{kH}{\otimes} kH_{(m)} \underset{kH}{\otimes} kP \underset{kP}{\otimes} kP_{(m)} \\ &\cong X \underset{kH}{\otimes} kH_{(m)} \underset{kH}{\otimes} kP_{(m)}\end{aligned}$$

Let

$$H = \langle g_2, g_3, \dots, g_n \rangle < \langle g_1, g_2, \dots, g_n \rangle = P$$

and $x_i = g_i - 1$. Then

$$\begin{aligned}\operatorname{rad}^m kH &= \left\langle \prod_{i=2}^{n} x_i^{s_i} \,\middle|\, 0 \leq s_i < p, \sum_{i=1}^{n} s_i \geq m \right\rangle \\ &< \left\langle \prod_{i=1}^{n} x_i^{s_i} \,\middle|\, 0 \leq s_i < p, \sum_{i=1}^{n} s_i \geq m \right\rangle \\ &= \operatorname{rad}^m kP\end{aligned}$$

and so the map

$$\begin{array}{rcl} kH_{(m)} \underset{kH}{\otimes} kP_{(m)} & \longrightarrow & kP_{(m)} \\ [h]\otimes[g] & \mapsto & [hg] \end{array}$$

is well-defined with inverse $[g] \mapsto 1\otimes[g]$. We therefore have that

$$\begin{aligned}\left(X_{(m)}\overset{P}{\uparrow}\right)_{(m)} &\cong X \underset{kH}{\otimes} kH_{(m)} \underset{kH}{\otimes} kP_{(m)} \\ &\cong X \underset{kH}{\otimes} kP_{(m)} \\ &\cong \left(X\overset{P}{\uparrow}\right)_{(m)}\end{aligned}$$

□

Proposition 6.16. $\mathcal{N}_P = \mathcal{M}_P$.

Proof. We aim to show that $\mathcal{N}_P$ satisfies the conditions defining $\mathcal{M}_P$.

It is easy to check that $\mathcal{N}_{C_p} = \mathcal{M}_{C_p}$ and so we proceed by induction on the rank of P with the additional assumption that each $X \in \mathcal{M}_P$ has simple head.

For an index p subgroup H of P we have by the induction hypothesis that each $X \in \mathcal{M}_H$ has simple head and so each $\mathfrak{r}^i\left(X\overset{P}{\uparrow}\right)$ also has simple head. This is enough to show that modules in $\mathcal{N}_P$ are indecomposable and hence by proposition 6.9 that $\mathcal{N}_P \subseteq \mathcal{M}_P$.

Since $kH \in \mathcal{M}_H$ we have that $kP \in \mathcal{N}_P$. Next we consider the restriction-induction property. Given $X \in \mathcal{N}_P \subseteq \mathcal{M}_P$ we know by proposition 6.10 that summands of $X\downarrow_L$ are in $\mathcal{M}_L$, we also have that there is an index p subgroup H of P with $L \le H < P$ and by proposition 6.9 summands of $X\downarrow_L\uparrow^H$ are in $\mathcal{M}_H$, thus we have that summands of $X\downarrow_L\uparrow^P$ are in $\mathcal{N}_P$.

Now we need only show that $\mathcal{N}_P$ is closed under taking quotients by powers of the radical. If m is the radical length of X then the radical length of $X\uparrow^P$ is $m + p - 1$ and thus lemma 6.15 tells us that

$$\mathfrak{r}^p\left(X\uparrow^P\right) = \left(X\uparrow^P\right)_{(m-1)} \cong \left((\mathfrak{r}X)\uparrow^P\right)_{(m-1)} = \mathfrak{r}^{p-1}\left((\mathfrak{r}X)\uparrow^P\right) \in \mathcal{N}_P$$

and similarly

$$\mathfrak{r}^{p+i}\left(X\uparrow^P\right) = \mathfrak{r}^{p-1}\left((\mathfrak{r}^{i+1}X)\uparrow^P\right) \in \mathcal{N}_P. \qquad \square$$

We can use proposition 6.16 to give a better bound for the global dimension we calculated in corollary 6.14. For instance we know that there are p modules in $\mathcal{M}_{C_p}$. Since C_p is the only isomorphism class of index p subgroups of $C_p \times C_p$ we can have at most p^2 radical length–dimension pairs in the class $\mathcal{M}_{C_p\times C_p}$ and therefore p^2 bounds the global dimension of $k[C_p \times C_p]$. This argument leads to the following corollary.

Corollary 6.17. *If $P = C_p \times \cdots \times C_p$ is an elementary abelian p-group and*

$$M = \bigoplus_{X\in\mathcal{M}_P} X$$

then

$$\operatorname{gl\,dim} \operatorname{End}_{kP}(M) \le |P|.$$

We may now use theorem 6.6 and separable equivalence to bound the global dimension of any group with elementary abelian Sylow p-subgroup. The proof of this is a direct consequence of corollary 6.7 and the discussion that follows it.

Theorem 6.18:

If G is a group with elementary abelian Sylow p-subgroup P, and k is an algebraically closed field of characteristic p then

$$\operatorname{rep\,dim} kG \le |P|.$$

We are able to perform explicit calculations to find the global dimension of $\mathrm{End}_{kP}(M)$ for some of the small p-groups (see section 6.4) and these show that the bound calculated here is not sharp.

6.3 Grading and filtration

In the last section we defined a class of modules for elementary abelian groups. Notice however that the definition of $\mathcal{M}_P$ makes sense for any p-group P and if the global dimension for the endomorphism ring is finite we can still use theorem 6.6 to bound the representation dimension of group algebras with P as a Sylow p-subgroup.

There is an equivalent definition of $\mathcal{M}_P$ for elementary abelian groups that uses the ideas of graded and filtered algebras. Again this definition can extend to general abelian groups however the two definitions are no longer equivalent at this level of generality. Having both definitions at our disposal will be useful for the calculations in section 6.4 and so we now devote some attention to graded and filtered algebras.

Definition 6.19 (Graded algebra)**.** Let A be a k-algebra. We say that A is a *graded algebra* (or $\mathbb{N}$-graded) if there is a vector space decomposition of A

$$A = \bigoplus_{n\in\mathbb{N}} A_n = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$$

such that for any $i, j \in \mathbb{N}$

$$A_i A_j \subseteq A_{i+j}.$$

If $x \in A_i$ then we say x is homogeneous of degree i and write $|x| = i$.

Definition 6.20 (Graded module)**.** Let $A = \bigoplus_i A_i$ be a graded algebra. An A-module is called a *graded module* if there is a vector space decomposition of M

$$M = \bigoplus_{n\in\mathbb{N}} M_n = M_0 \oplus M_1 \oplus M_2 \oplus \cdots$$

such that for any $i, j \in \mathbb{N}$

$$M_i A_j \subseteq M_{i+j}.$$

A homomorphism $f\colon M \to N$ between graded modules is called a graded homomorphism if for all $i \in \mathbb{N}$ we have

$$f(M_i) \subseteq N_i$$

Definition 6.21 (Filtered algebra)**.** Let A be a k-algebra. We say that A is a *filtered algebra* if there is a sequence of vector spaces

$$A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$$

such that for any $i, j \in \mathbb{N}$

$$A_i A_j \subseteq A_{i+j}.$$

Given a module M for a filtered algebra A, we automatically have a filtration on M given by

$$\begin{array}{ccccccc} M = M_0 & \supseteq & M_1 & \supseteq & M_2 & \supseteq & \cdots \\ & & \wr\parallel & & \wr\parallel & & \\ & & MA_1 & & MA_2 & & \end{array}$$

and so we don't need a separate definition for a filtered module in this instance.

Notice that if $A = \bigoplus A_i$ is a graded algebra then we can define a filtered algebra B from the grading via

$$B_i = \bigoplus_{j \geq i} A_j$$

We will fix a standard filtration on an abelian p-group P. Let P be given by the product of cyclic groups

$$P = C_{p^{r_1}} \times C_{p^{r_2}} \times \ldots \times C_{p^{r_n}}$$

where g_i is a generator for the i-th cyclic group, so that we have an isomorphism

$$\begin{array}{ccc} \dfrac{k[x_1, x_2, \ldots, x_n]}{(x^{p^{r_1}}, x^{p^{r_2}}, \ldots, x^{p^{r_n}})} & \xrightarrow{\sim} & kP \\ x_i & \mapsto & g_i - 1 \end{array}$$

We can impose a grading on the polynomial ring by selecting degrees for each of the x_i and this induces a grading on kP. If p^N is the exponent of P then we set

$$|x_i| = \frac{p^N}{o(g_i)}$$

where $o(g)$ represents the order of the group element g. Thus if g_i has maximal order in the group then $(g_i - 1)$ is homogeneous of degree 1.

Of course due to the remark above this grading induces a filtration on the group algebra. The following proposition demonstrates that the filtration induced is independent of the choice of generators used to define the grading.

Proposition 6.22. *Let $P = C_{p^{r_1}} \times \cdots \times C_{p^{r_n}}$ be an abelian p-group of exponent p^N.*

Let $\{g_1, \ldots, g_n\}$ be a set of generators satisfying $o(g_i) = p^{r_i}$.

Let $\{h_1, \ldots, h_n\}$ be a second set of generators satisfying $o(h_i) = p^{r_i}$.

Let $A \cong kP$ be the graded algebra with grading generated by $|g_i - 1| = p^{N-r_i}$.

Let $B \cong kP$ be the graded algebra with grading generated by $|h_i - 1| = p^{N-r_i}$.

In this setting the filtration on A equals that on B: that is for each $i \in \mathbb{N}$

$$A_{\geq i} = \bigoplus_{j \geq i} A_i = \bigoplus_{j \geq i} B_i = B_{\geq i}$$

Proof. We may assume that $r_i \leq r_{i+1}$ and that

$$h_i = \prod_{j=1}^{n} g_j^{u_{ij}}$$

for some values u_{ij}. Note that since the orders of g_i and h_i coincide we must have that $p^{r_j - r_i}$ divides u_{ij}.

We wish to show that when $i \in \{1, 2, \ldots, n\}$ then $(h_i - 1) \in A_{\geq p^{r_n - r_i}}$.

It is a simple exercise to show that

$$h_i - 1 = \prod_{j=1}^{n} g_j^{u_{ij}} = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} \prod_{j \in I} (g_j^{u_{ij}} - 1)$$

thus we need only show that

$$g_j^{u_{ij}} - 1 \in A_{\geq p^{r_n - r_i}}$$

for all j. We can write

$$g_j^{u_{ij}} - 1 = (g_j - 1)^{u_{ij}} + u_{ij}(g_j - 1)^{u_{ij}-1} + \binom{u_{ij}}{2}(g_j - 1)^{u_{ij}-2} + \cdots + u_{ij}(g_j - 1).$$

If $r_j > r_i$ then p divides all terms apart from the first and so

$$|g_j^{u_{ij}} - 1| = u^{ij} p^{r_n - r_j} \geq p^{r_j - r_i} p^{r_n - r_j} \geq p^{r_n - r_i}$$

as needed. On the other hand if $r_j \leq r_i$ then all terms in the sum are in

$$A_{\geq p^{r_n - r_j}} \subseteq A_{\geq p^{r_n - r_i}}.$$

As there is nothing to differentiate between A and B we can reverse the argument to show the opposite inclusion to complete the result. □

Example. We demonstrate with an example that even though the filtrations are the same the gradings need not be. Consider the group

$$C_2 \times C_2 = \langle g, h \rangle$$

We can calculate one grading using the generators g and h, and a second grading using the generators gh and h. The element $gh - 1$ is homogeneous of degree 1 using the second grading but since

$$gh - 1 = (g-1)(h-1) + (g-1) + (h-1)$$

it is clearly not homogeneous using the first.

In the case that $P = C_p \times \cdots \times C_p$ is a rank n elementary abelian group generated by $\{g_1, \dots, g_n\}$ then

$$kP_{\geq m} = \left\langle \prod_{i=1}^{n} (g_i - 1)^{s_i} \;\middle|\; 0 \leq s_i < p,\ \sum_{i=1}^{n} s_i \geq m \right\rangle = \operatorname{rad}^m kP.$$

We can give an alternative to definition 6.8:

Definition 6.23 ($\mathcal{M}_P$)**.** Let P be an elementary abelian p-group and k a field of characteristic p. Let $\mathcal{M}_P$ be the set of indecomposable kP-modules that is minimal with respect to the following properties:

(a) $kP \in \mathcal{M}_P$;

(b) if $X \in \mathcal{M}_P$, H is a subgroup of P and Z is an indecomposable summand of $X{\downarrow_H}{\uparrow^P}$ then $Z \in \mathcal{M}_P$;

(c) if $X \in \mathcal{M}_P$, d is a positive integer and Z is an indecomposable summand of $X/X_{\geq d}$ then $Z \in \mathcal{M}_P$.

We now have two alternative yet equivalent definitions of $\mathcal{M}_P$ for elementary abelian groups that generalise in different ways to general abelian p-groups. The first of these definitions can also generalise to non-abelian p-groups.

When generalising to groups that are not elementary abelian we will use the notation $\mathcal{R}_P$ for the class as described in definition 6.8 and $\mathcal{F}_P$ for the class described in definition 6.23.

6.4 Calculations

In section 6.2 we calculated an upper bound for the representation dimension of group algebras for groups with elementary abelian Sylow p-subgroups. For some small p-groups we can calculate directly the global dimension for certain generators and use the theorems of that section to give more precise bounds for the representation dimension.

We have an algebra A and a generator-cogenerator M_A and wish to calculate minimal projective resolutions for $E = \operatorname{End} M_A$. In order to perform these calculations we will use a correspondence between the projective modules for E and summands of M_A. As additive categories we have an equivalence

$$\begin{array}{rcl} \operatorname{proj} E & \overset{\sim}{\longleftrightarrow} & \operatorname{add} M_A \\ X & \mapsto & X \underset{E}{\otimes} M_A \\ \operatorname{Hom}_A(M, N) & \leftarrow & N \end{array}$$

To see that these give an inverse equivalence we must check that the compositions

$$X_E \mapsto \operatorname{Hom}_A(M, X \underset{E}{\otimes} M) \qquad N_A \mapsto \operatorname{Hom}_A(M, N) \underset{E}{\otimes} M$$

are naturally isomorphic to the identity functors. If $X = E$ then we have the isomorphism

$$\begin{array}{rcl} \operatorname{Hom}_A(M, E \otimes M) & \overset{\sim}{\longrightarrow} & \operatorname{Hom}_A(M, M) \\ [m \mapsto \phi_m \otimes \theta(m)] & \mapsto & \phi_m \circ \theta \end{array}$$

and via the additivity of the functors (and the fact that projectives are summands of free modules) this gives the first natural isomorphism. Similarly we have an isomorphism

$$\begin{array}{rcl} \operatorname{Hom}_A(M, M) \underset{E}{\otimes} M & \overset{\sim}{\longrightarrow} & M \\ \phi \otimes m & \mapsto & \phi(M) \end{array}$$

and via additivity this gives the natural isomorphism required for the second composition.

Let X_E be an $\operatorname{End} M_A$ module and

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

be a projective resolution of M. The correspondence detailed above then gives us a complex of modules in $\operatorname{add} M$

$$\cdots \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0$$

For any $i > 0$ we know that

$$0 \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow P_{i-1} \longrightarrow 0$$

is exact and this is true if and only if

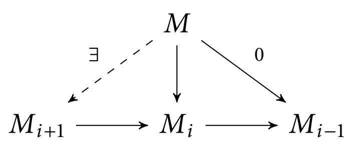
$$0 \longrightarrow \operatorname{Hom}_E(Q, P_{i+1}) \longrightarrow \operatorname{Hom}_E(Q, P_i) \longrightarrow \operatorname{Hom}_E(Q, P_{i-1}) \longrightarrow 0$$

is exact for all projective E-modules Q. This is then true if and only if

$$0 \longrightarrow \operatorname{Hom}_A(N, M_{i+1}) \longrightarrow \operatorname{Hom}_A(N, M_i) \longrightarrow \operatorname{Hom}_A(N, M_{i-1}) \longrightarrow 0$$

is exact for all modules $N \in \operatorname{add} M$.

Given this argument the property of exactness of the projective resolution P_* translates to the property that any A-module map from M to the kernel of $M_i \to M_{i-1}$ factors through M_{i+1} whenever $i > 0$.



Now if X is simple and the resolution is minimal so that $P_0 \to X$ is a projective cover, then exactness at P_0 means that P_1 maps onto the radical of P_0. This is true if and only if for every non-split map from an indecomposable projective E-module factors through P_1. Again this can be transferred through the correspondence: we have that M_0 is an indecomposable summand of M and every non-split map from an indecomposable summand of M factors through the module M_1.

In order to calculate the minimal projective resolutions of simple $\operatorname{End} M_A$ modules we may begin with M_0, the corresponding indecomposable summand of M. We then calculate the minimal object M_1 of $\operatorname{add} M$ for which any non-split map from M to M_0 factors through M_1. Once we have calculated the map $M_i \to M_{i-1}$ we then calculate M_{i+1} as the minimal object of $\operatorname{add} M$ for which any map from M to the kernel of $M_i \to M_{i-1}$ factors through M_{i+1}. In this way we may calculate the length of projective resolutions.

Using the *Magma* software we are able to use this approach to calculate the projective resolutions for some low order abelian p-groups. The code used can be found in appendix A and for the abelian groups the resolutions themselves can be found in appendix B. For the non-abelian groups we use the *GAP* notation G_o^i to identify the groups.

By definition the sets $\mathcal{R}_P$ and $\mathcal{F}_P$ are closed under the (res-ind) property. In order to bound the representation dimension of any group G with P as a Sylow p-subgroup we must also check that the set $\mathcal{R}_P$ or $\mathcal{F}_P$ is closed under the (isom) property. This again can be checked using *Magma* and we find that this property holds for all the groups in the tables below except for G_{16}^3, G_{16}^4 and G_{16}^{11}. See the comments regarding G_{16}^3 and G_{16}^{11} below.

Abelian 2-groups

P	C_2	C_4	$C_2 \times C_2$	C_8	$C_4 \times C_2$	$C_2 \times C_2 \times C_2$
rep dim $G \leq$	2	2	3	2	5	6

P	C_{16}	$C_8 \times C_2$	$C_4 \times C_4$	$C_4 \times C_2 \times C_2$	$C_2 \times C_2 \times C_2 \times C_2$
rep dim $G \leq$	2	5	6	8	8

Abelian 3-groups

P	C_3	C_9	$C_3 \times C_3$
rep dim $G \leq$	2	2	5

Non-abelian 2-groups

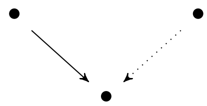
P	$D_4 = G_8^3$	$Q_8 = G_8^4$
rep dim $G \leq$	4	5

P	$(C_4 \times C_2) \rtimes C_2 = G_{16}^3$	$C_4 \rtimes C_4 = G_{16}^4$	$C_8 \rtimes C_2 = G_{16}^6$
rep dim $G \leq$	?	?	10

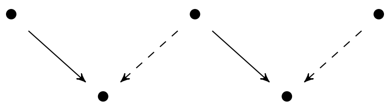
P	$D_8 = G_{16}^7$	$QD_{16} = G_{16}^8$	$Q_{16} = G_{16}^9$	$D_4 \times C_2 = G_{16}^{11}$
rep dim $G \leq$	4	5	5	?

P	$Q_8 \times C_2 = G_{16}^{12}$	$(C_4 \times C_2) \rtimes C_2 = G_{16}^{13}$
rep dim $G \leq$	7	7

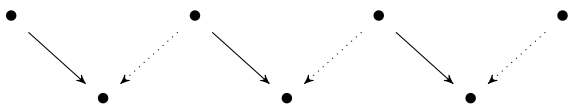
Notice that for the groups G_{16}^3 and G_{16}^{11} we have not calculated a bound. In these cases the class of modules $\mathcal{R}_G$ appears to be infinite. The group G_{16}^3 can be interpreted as the subgroup of S_8 generated by the permutations $g = (1,2,3,4)(5,6,7,8)$ and $h = (1,5)(3,7)$. $\mathcal{R}_G$ contains a module that on restriction to the subgroup $H = \langle g^2, h\rangle$ can be represented as

$X_1 =$ 

where the dotted arrow represents the action of $g^2 - 1$ and the solid arrow represents the action of $h - 1$. Now if we induce to G, take a quotient by the cube of the radical and restrict to the subgroup $L = \langle ghgh, h\rangle$ we obtain the module

$X_2 =$ 

as a summand of $\left.\dfrac{X_1\uparrow^G}{\operatorname{rad}^3 X_1\uparrow^G}\right\downarrow_L$, where the dashed arrow represents the action of $ghgh-1$.
Continuing in this manner we get

$X_3 =$ 

as a summand of $\left.\dfrac{X_2\uparrow^G}{\operatorname{rad}^3 X_2\uparrow^G}\right\downarrow_H$ and through *Magma* calculations we can see that the pattern continues at least until the module X_{40}.

A Magma code

Included in this appendix is the code used for the calculations in chapter 6. We will give an example of how to calculate a projective resolution as an illustration of how to use these functions.

We will begin with the group $G = C_4 \times C_2$ and calculate the class of modules $\mathcal{F}_G$. We will then calculate the resolution for the projective module corresponding to the regular kG-module in $\mathcal{F}_G$.

Example.

```
G := AbelianPGroup(2, [2,1]); G;
     Permutation group G acting on a set of cardinality 6
     Order = 8 = 2^3
         (1, 2, 3, 4)
         (5, 6)

M := GetM_filt(G); M;
     [
         GModule of dimension 1 over GF(2),
         GModule of dimension 2 over GF(2),
         GModule of dimension 2 over GF(2),
         GModule of dimension 2 over GF(2),
         GModule of dimension 3 over GF(2),
         GModule of dimension 3 over GF(2),
         GModule of dimension 3 over GF(2),
         GModule of dimension 4 over GF(2),
         GModule of dimension 4 over GF(2),
         GModule of dimension 4 over GF(2),
         GModule of dimension 4 over GF(2),
         GModule of dimension 5 over GF(2),
         GModule of dimension 6 over GF(2),
         GModule of dimension 6 over GF(2),
         GModule of dimension 6 over GF(2),
         GModule of dimension 7 over GF(2),
         GModule of dimension 8 over GF(2)
     ]

f := MinimalResolution(M[17], M); ShowResolution(f, M);
```

```
[ 1  2  5  6  8  9 10 11 12 13 14 15 16 17]
[ 1  2  3  3  4  4  4  4  5  6  6  6  7  8]
[ 0  0  0  0  0  0  0  0  0  0  0  0  0  1]
[ 0  0  0  0  1  1  0  0  0  0  1  1  0  0]
[ 0  0  1  1  0  0  1  0  0  1  0  0  1  0]
[ 0  1  0  0  0  0  0  1  1  0  0  0  0  0]
[ 1  0  0  0  0  0  0  0  0  0  0  0  0  0]
```

The final output is a table giving the number of copies of each $\mathcal{F}_G$ module in each term of the projective resolution. The first row of the table refers to the index of the module. The second row indicates the dimension of the summand. The third and successive rows detail the decomposition of each term in the resolution.

Functions

```
AbelianPGroup := function(p, orderseq);
  if not Type(orderseq) eq SeqEnum then
    orderseq := [ orderseq ];
  end if;
  error if not IsPrime(p), Sprintf("%o is not prime.",p);
  if #orderseq eq 0 then
    return CyclicGroup(1);
  end if;
  error if Type(orderseq[1]) ne RngIntElt,
    "orderseq must contain Integers";
  return [ DirectProduct( [CyclicGroup(p^i) :
            i in Sort(orderseq,func<x,y|y-x>) ])][1];
end function;

AppendSummandsIfNotIso := procedure(~M, N);
  if IsZero(N) then return; end if;
  for Summand in IndecomposableSummands(N) do
    add := true;
    for X in M do
      if IsIsomorphic(Summand, X) then
        add := false;
        break X;
      end if;
    end for;
    if add then
      Append(~M, Summand);
    end if;
  end for;
end procedure;

IdxSeq := function(maxvals);
  idxseq := [];
  idx := [ 1 : i in maxvals ];
```

```
  idx[1] := 0;
  repeat
    i := 1;
    while idx[i] eq maxvals[i] do
      i := i + 1;
    end while;
    for j in [1..i-1] do
      idx[j] := 1;
    end for;
    idx[i] := idx[i] + 1;
    Append(~idxseq, idx);
  until idx eq maxvals;
  return [ [ j-1 : j in i ] : i in idxseq ];
end function;

Head := function(X);
  RL,V := RadicalLayers(X);
  return [ X!V[i] : i in [1..RL[1]-RL[2]] ];
end function;

GroupMonomial := function(N, degs);
  G := Group(N);
  error if #degs ne #Generators(G),
        "Length of sequence invalid.";
  m := Head(N);
  for i in [1..#degs] do
    for j in [1..degs[i]] do
      m := [ x*G.i - x : x in m ];
    end for;
  end for;
  return m;
end function;

FilteredLayers := function(N);
  G := Group(N);
  gens := [ G.i : i in [1..#Generators(G)] ];
  maxvals := [ Order(g) : g in gens ];
  degrees  := [ Max(maxvals)/x : x in maxvals ];
  m := 0;
  idxseq := IdxSeq(maxvals);
  FL := [];
  repeat
    mons := [ mon : mon in GroupMonomial(N,idx),
            idx in idxseq | &+[ idx[i]*degrees[i] :
                i in [1..#degrees] ] ge m ];
    J := sub<N | mons>;
    Append(~FL, Dimension(J));
    m := m + 1;
  until IsZero(J);
  return FL;
end function;
```

```
AppendSummandsOfFiltQuotients := procedure(~M, N);
  G := Group(N);
  if #Generators(G) eq 0 then
    return;
  end if;
  gens := [ G.i : i in [1..#Generators(G)] ];
  maxvals := [ Order(g) : g in gens ];
  degrees  := [ Max(maxvals)/x : x in maxvals ];
  m := 1;
  idxseq := IdxSeq(maxvals);
  repeat
    mons := [ mon : mon in GroupMonomial(N,idx),
                idx in idxseq | &+[ idx[i]*degrees[i] :
                    i in [1..#degrees] ] ge m ];
    J := sub<N | mons>;
    AppendSummandsIfNotIso(~M, quo<N|J>);
    m := m + 1;
  until IsZero(J);
end procedure;

AppendSummandsOfRadQuotients := procedure(~M, N);
    J := JacobsonRadical(N);
    while not IsZero(J) do
        AppendSummandsIfNotIso(~M, quo<N | J>);
        J := JacobsonRadical(J);
    end while;
    AppendSummandsIfNotIso(~M, N);
end procedure;

CloseUnderResInd := procedure(~M);
  G := Group(M[1]);
  repeat
    numM := #M;
    for rec in Subgroups(G) do
      H := rec`subgroup;
        for k in [1..numM] do
          N := M[k];
          for R in IndecomposableSummands(
                       Restriction(N, H)) do
            for RI in IndecomposableSummands(
                           Induction(R, G)) do
              AppendSummandsIfNotIso(~M, RI);
            end for;
          end for;
        end for;
    end for;
  until numM eq #M;
end procedure;

CloseUnderFiltQuotients := procedure(~M);
```

```
  repeat
    numM := #M;
    for N in M do
      AppendSummandsOfFiltQuotients(~M, N);
    end for;
  until numM eq #M;
end procedure;

CloseUnderRadQuotients := procedure(~M);
    repeat
        numM := #M;
        for N in M do
            AppendSummandsOfRadQuotients(~M, N);
        end for;
    until numM eq #M;
end procedure;

IsGrpAutMod := function(X,Y);
  if Group(X) ne Group(Y) then
    return false;
  end if;
  if Dimension(X) ne Dimension(Y) then
    return false;
  end if;
  if RadicalLayers(X) ne RadicalLayers(Y) then
    return false;
  end if;
  G := Group(X);
  if IsIsomorphic(X,Y) then return true,
       [ G.i : i in [1..#Generators(G) ] ];
  end if;
  auts := [ [ g : g in G | Order(g) eq Order(G.i) ] :
               i in [1..#Generators(G)] ];
  idx := [ 1 : i in auts ];
  maxidx := [ #aut : aut in auts ];
  idx[1] := 0;
  repeat
    i := 1;
    while idx[i] eq maxidx[i] do
      i := i + 1;
    end while;
    for j in [1..i-1] do
      idx[j] := 1;
    end for;
    idx[i] := idx[i]+1;
    if sub<G| [ auts[i][idx[i]] :
                 i in [1..#idx] ]> eq G then
      tAct := [
        Matrix([ X.i*auts[j][idx[j]]
          : i in [1..Dimension(X)] ]) : j in [1..#idx] ];
      tX := GModule(G, tAct);
```

```
      if IsIsomorphic(tX, Y) then
        return true, [ auts[i][idx[i]] : i in [1..#idx] ];
      end if;
    end if;
  until maxidx eq idx;
  return false;
end function;

CollectAutMods := procedure(~M);
  autmods := [];
  idx := { 1..#M };
  while #idx gt 0 do
    i := Random(idx);
    Append(~autmods, []);
    for j in [1..#M] do
      if IsGrpAutMod(M[i], M[j]) then
        Append(~autmods[#autmods], j);
        Exclude(~idx, j);
      end if;
    end for;
  end while;
  Sort(~autmods);
  revautmods := [];
  for i in [1..#autmods] do
    for j in autmods[i] do
      revautmods[j] := i*#M*2+j;
    end for;
  end for;
  ParallelSort(~revautmods, ~M);
end procedure;

SortM := procedure(~M);
  Sort(~M, func<x,y|Dimension(x)-Dimension(y)>);
  CollectAutMods(~M);
end procedure;

GetM_filt := function(G);
  error if Type(G) ne GrpPerm,
    Sprintf("Expected GrpPerm, given %o",Type(G));
  error if not IsAbelian(G),
    Sprintf("%o is not abelian.", G);
  p := PrimeDivisors(Order(G));
  error if #p gt 1,
    Sprintf("%o is not a p-group.", G);

  if #p lt 1 then
    p := 2;
  else
    p := p[1];
  end if;
  RM := PermutationModule(G, sub<G|>, GF(p));
```

```
  M := [ ];
  AppendSummandsIfNotIso(~M, RM);
  repeat
    numM := #M;
    CloseUnderResInd(~M);
    CloseUnderFiltQuotients(~M);
  until numM eq #M;

  SortM(~M);

  return M;
end function;

GetM_rad := function(G);
  error if Type(G) ne GrpPerm,
    Sprintf("Expected GrpPerm, given %o",Type(G));
  p := PrimeDivisors(Order(G));
  error if #p gt 1,
    Sprintf("%o is not a p-group.", G);

  if #p lt 1 then
    p := 2;
  else
    p := p[1];
  end if;
  RM := PermutationModule(G, sub<G|>, GF(p));

  M := [ ];
  AppendSummandsIfNotIso(~M, RM);
  repeat
    numM := #M;
    CloseUnderResInd(~M);
    CloseUnderRadQuotients(~M);
  until numM eq #M;

  SortM(~M);

  return M;
end function;

SumMaps := function(m);
  D  := [ Domain(x) : x in m ];
  CD := Codomain( m[1] );
  error if not &and[ IsIsomorphic(CD, Codomain(x))
        : x in m ], "Codomains do not match.";
  D,incl,proj := DirectSum(D);
  f := MapToMatrix(proj[1])*m[1];
  for i in [2..#m] do
    f := f + MapToMatrix(proj[i])*m[i];
  end for;
```

```
  return f,incl,proj;
end function;

QtoNFactorsThroughPsToN := function(QtoN, PsToN);
  Q := Domain(QtoN);
  N := Codomain(QtoN);
  FM := [];
  for i in [1..#PsToN] do
    PtoN := PsToN[i];
    P := Domain(PtoN);
    B := Basis(GHom(Q, P));
    for v in B do
      Append(~FM, v*PtoN);
    end for;
  end for;
  SubGHom := sub<KMatrixSpace(Field(Q),
                 Dimension(Q), Dimension(N)) | FM>;
  return QtoN in SubGHom;
end function;

RemoveFactoringMaps := function(d);
  for i in Reverse([1..#d]) do
    testD := Remove(d, i);
    m := d[i];
    if QtoNFactorsThroughPsToN(m, testD) then
      Remove(~d, i);
    end if;
  end for;
  return d;
end function;

Cover := function( N, canBeSplit, M );
  d := [**];
  maxdim := 0;
  for i in [1..#M] do
    if Dimension(M[i]) gt maxdim then
      maxdim := Dimension(M[i]);
    end if;
  end for;
  for coRank in [0..maxdim-1] do
    for dimOfMod in Reverse([coRank+1..maxdim]) do
      appended := false;
      for i in [1..#M] do
        X := M[i];
        if Dimension(X) eq dimOfMod then
          if canBeSplit or (not IsIsomorphic(X, N)) then
            B := Basis(GHom(X,N));
            for m in B do
              if Rank(m) eq dimOfMod-coRank then
                if #d eq 0 then
                  Append(~d, m);
```

```
              else
                if not QtoNFactorsThroughPsToN(m, d) then
                  Append(~d, m);
                  appended := true;
                end if;
              end if;
            end if;
          end for;
        end if;
      end if;
    end for;
    if appended then
      d := RemoveFactoringMaps(d);
    end if;
  end for;
end for;
return d;
end function;

MinimalResolution := function(N, M);
  f := [**];
  K := [**];

  n := 0;
  d := [* GHom(N, N)!0 *];
  while true do
    n := n + 1;
    f[n] := SumMaps(d);

    K[n] := Kernel(f[n]);

    if (IsZero(K[n])) then
      print "Resolution complete";
      return f;
    end if;

    print "Calculating cover [",n,"]...";
    d := Cover(K[n], n gt 1, M);
  end while;
end function;

ShowResolution := function(f, M);
  Res := [];
  res := [ 0 : i in [1..#M] ];
  nonz := {};
  for m in f do
    NumSeq := [ 0 : i in [1..#M] ];
    for X in DirectSumDecomposition(Domain(m)) do
      for i in [1..#M] do
        if IsIsomorphic(X, M[i]) then
          NumSeq[i] := NumSeq[i] + 1;
```

```
          Include(~nonz, i);
          break i;
        end if;
      end for;
    end for;
    Append(~Res, NumSeq);
  end for;
  nonz := Sort(SetToSequence(nonz));
  T := ZeroMatrix(Integers(), #Res+2, #nonz);
  for j in [1..#nonz] do
    T[1][j] := nonz[j];
    T[2][j] := Dimension(M[nonz[j]]);
    for i in [1..#Res] do
      T[i+2][j] := Res[i][nonz[j]];
    end for;
  end for;
  return T;
end function;

IsomorphismClassesOfSubgroups := function(G);
  ICs := [];
  for rec in Subgroups(G) do
    H := rec`subgroup;
    appended := false;
    for i in [1..#ICs] do
      if IsIsomorphic(H, ICs[i][1]) then
        Append(~ICs[i], H);
        appended := true;
        break i;
      end if;
    end for;
    if not appended then
      Append(~ICs, [ H ] );
    end if;
  end for;
  return ICs;
end function;

RestrictionOfM := function(M, H);
  Mres := [];
  for N in M do
    AppendSummandsIfNotIso(~Mres, Restriction(N, H));
  end for;
  return Mres;
end function;

ModuleThroughIsomorphism := function(X, phi);
  H := Domain(phi);
  L := Codomain(phi);
  error if Group(X) ne L,
    "X is not a module for the codomain of phi.";
```

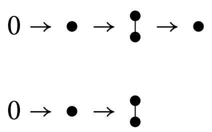
```
  phiX := GModule(H,
    [ Matrix([ X.i*phi(H.j) : i in [1..Dimension(X)] ])
      : j in [1..#Generators(H)] ] );
  return phiX;
end function;

RestrictionToSubgroupsAreIsomorphic := function(M, H, L);
  resH := RestrictionOfM(M, H);
  resL := RestrictionOfM(M, L);
  isI, phi := IsIsomorphic(H, L);
  if not isI then
    return false;
  end if;
  for N in resL do
    phiN := ModuleThroughIsomorphism(N, phi);
    if not &or[ IsIsomorphic(phiN, X) : X in resH ] then
      return false;
    end if;
  end for;
  return true;
end function;

RestrictionToIsomorphicSubgroupsIsValid := function(M);
  G := Group(M[1]);
  ICs := IsomorphismClassesOfSubgroups(G);
  for IC in ICs do
    if #IC gt 1 then
      for Pair in CartesianProduct(IC,IC) do
        H := Pair[1];
        L := Pair[2];
        if H ne L then
          if not RestrictionToSubgroupsAreIsomorphic(M, H, L) then
            return false;
          end if;
        end if;
      end for;
    end if;
  end for;
  return true;
end function;
```

B Calculated resolutions

We represent modules by diagrams with a vertex for each basis element of the module and edges representing the action of the algebra.

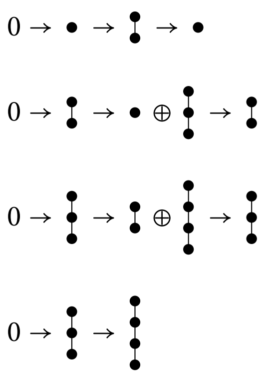
B.1 C_2

If $C_2 = \langle g \rangle$ then the module (two vertices joined by an edge) is the regular module and the edge represents the action of $g - 1$.

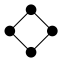


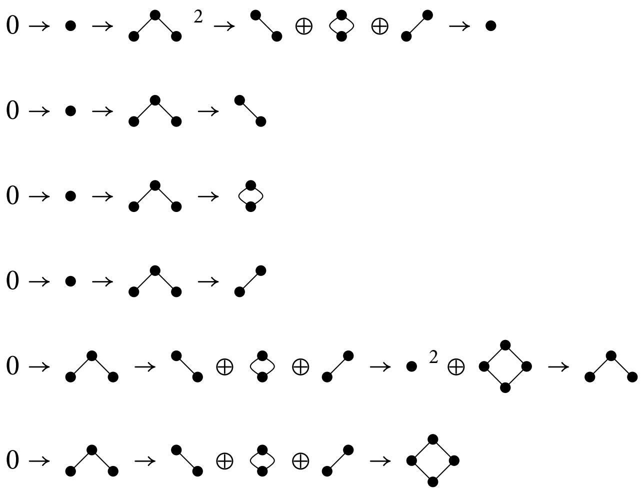
B.2 C_4

If $C_4 = \langle g \rangle$ then each edge represents the action of $g - 1$.

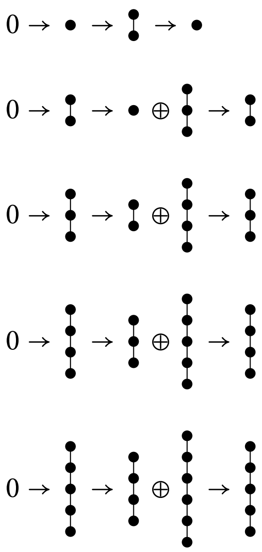


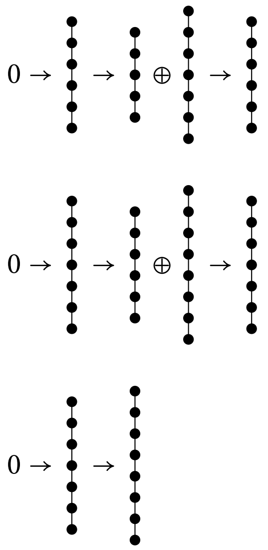
B.3 $C_2 \times C_2$

If $C_2 \times C_2 = \langle g, h \rangle$ then the module  is the regular module. Each edge  represents the action of $g - 1$ and each edge  represents the action of $h - 1$. The notation  shows that g and h act in the same way.

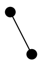
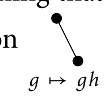


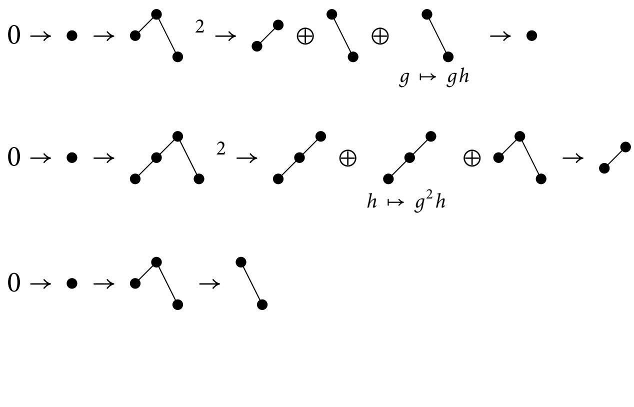
B.4 C_8

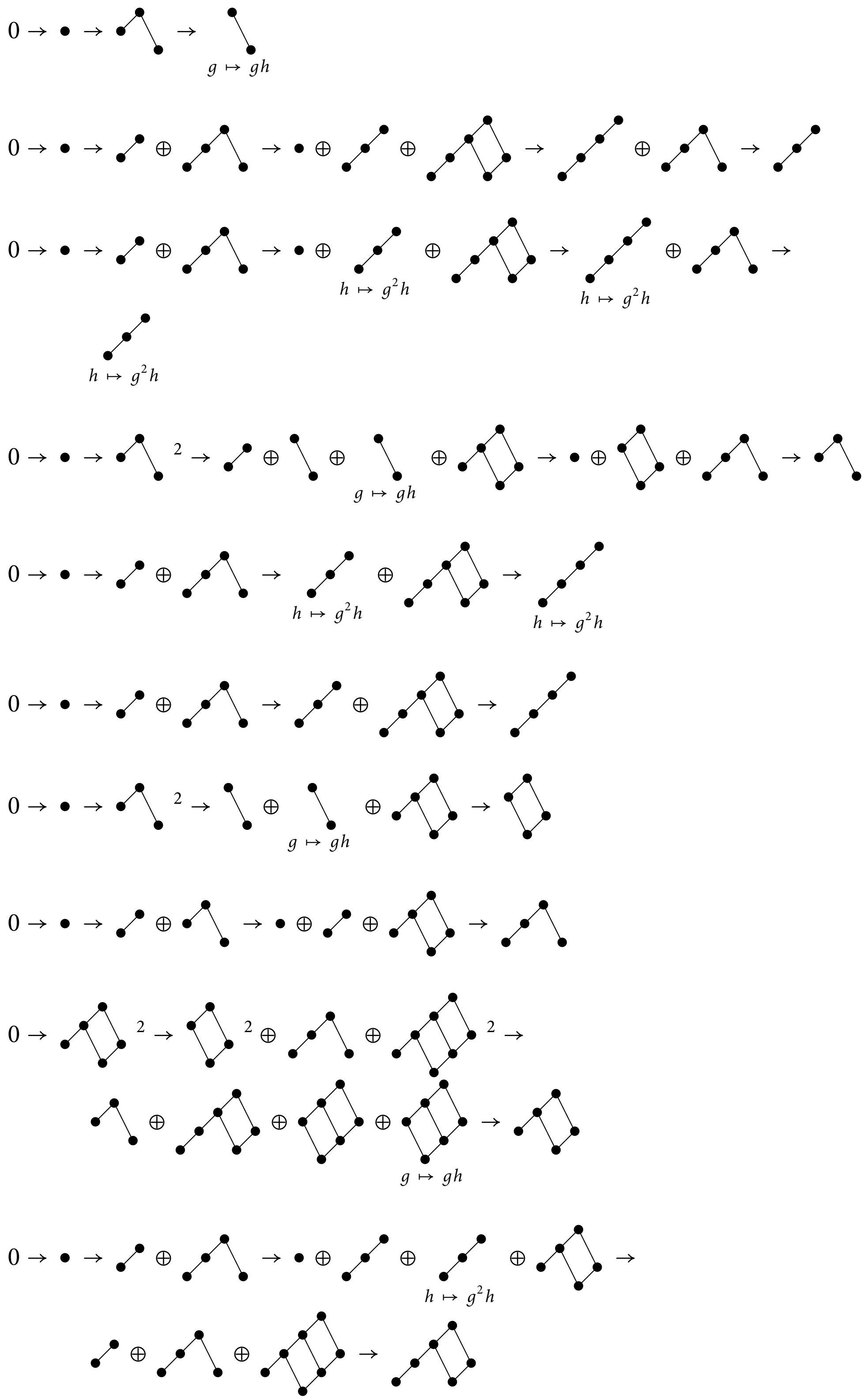




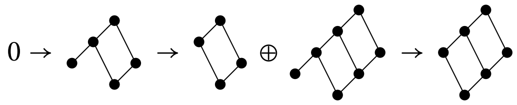
B.5 $C_4 \times C_2$

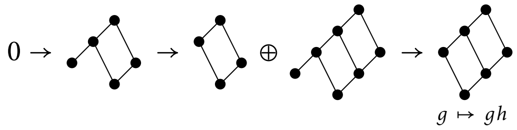
As $C_4 \times C_2$ is a product of non-isomorphic cyclic groups there is a choice of which module to use for the calculation of a bound for representation dimension. Here we use the class $\mathcal{F}_G$, which is based on the filtered algebra approach. The diagrams are drawn compatible with the degrees in this filtration. If $C_4 \times C_2 = \langle g, h \rangle$ with $g^4 = h^2 = 1$ then  represents the action of $g-1$ and  represents the action of $h-1$. In some cases the diagrams are denoted with a group automorphism, meaning that the diagram is drawn for a different basis of kG. For example the the notation  represents the 2-dimensional module where the action of h is as per the diagram, whilst the action of g on the module is the action of gh on the diagram.

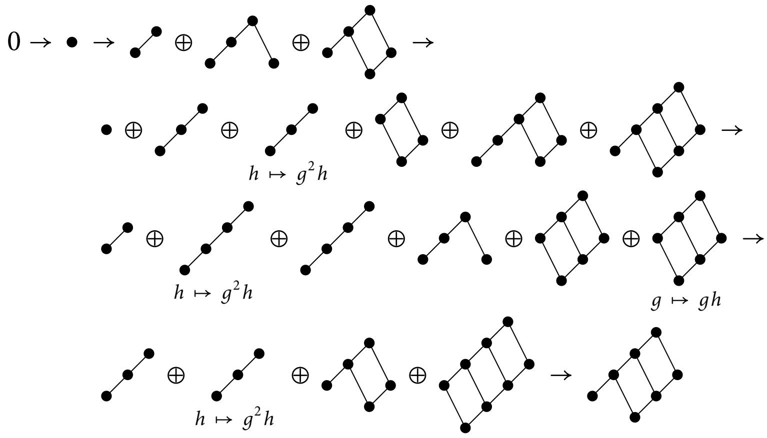


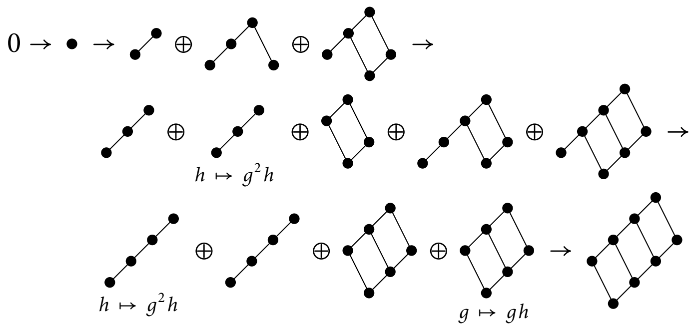


g ↦ gh
h ↦ g²h
h ↦ g²h
h ↦ g²h
g ↦ gh
h ↦ g²h
h ↦ g²h
g ↦ gh
g ↦ gh
h ↦ g²h



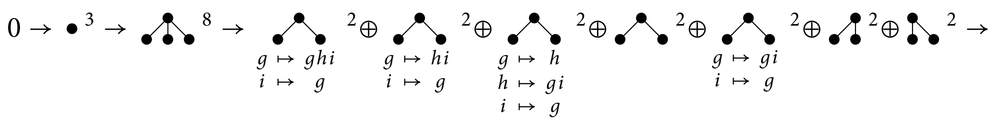


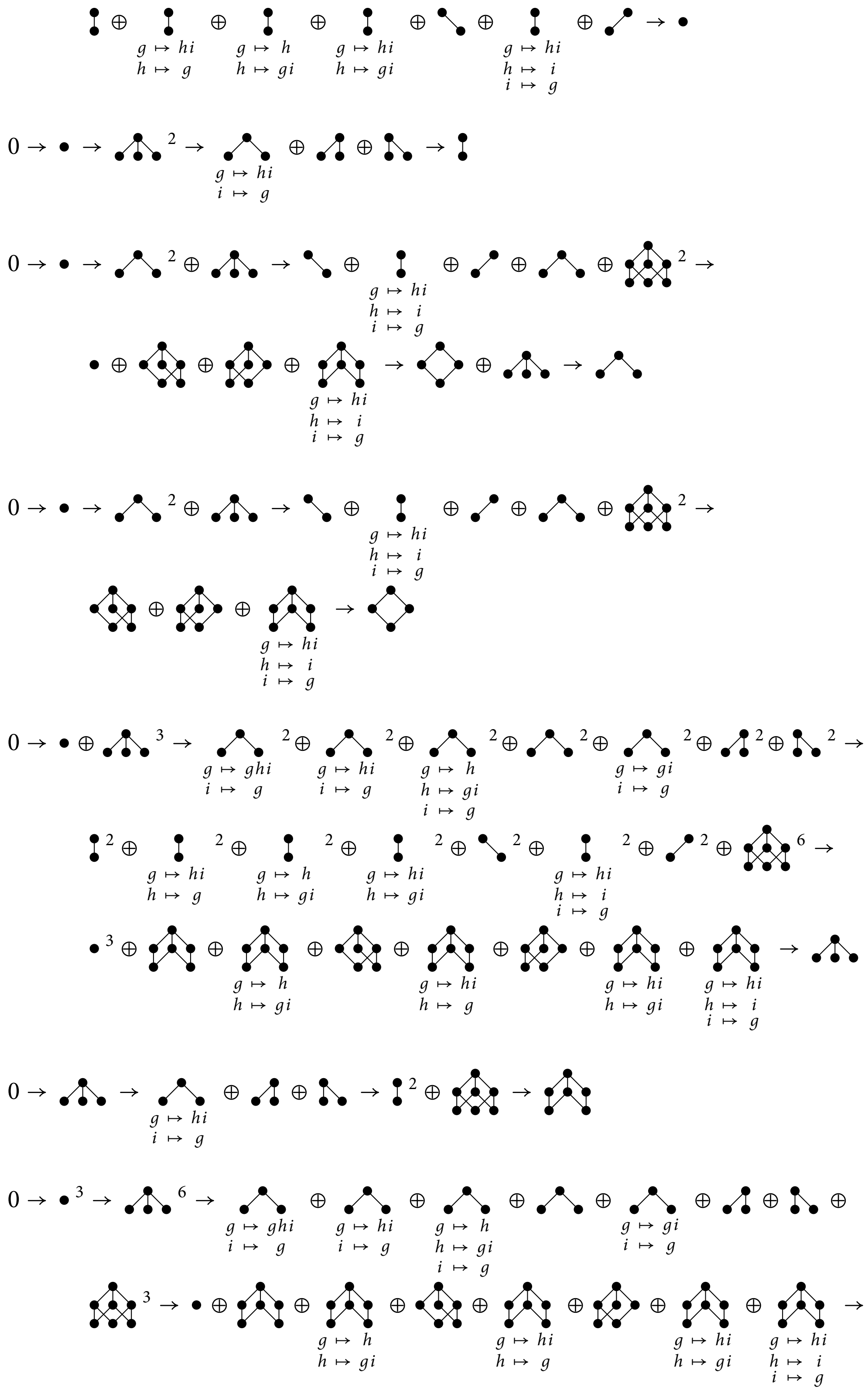


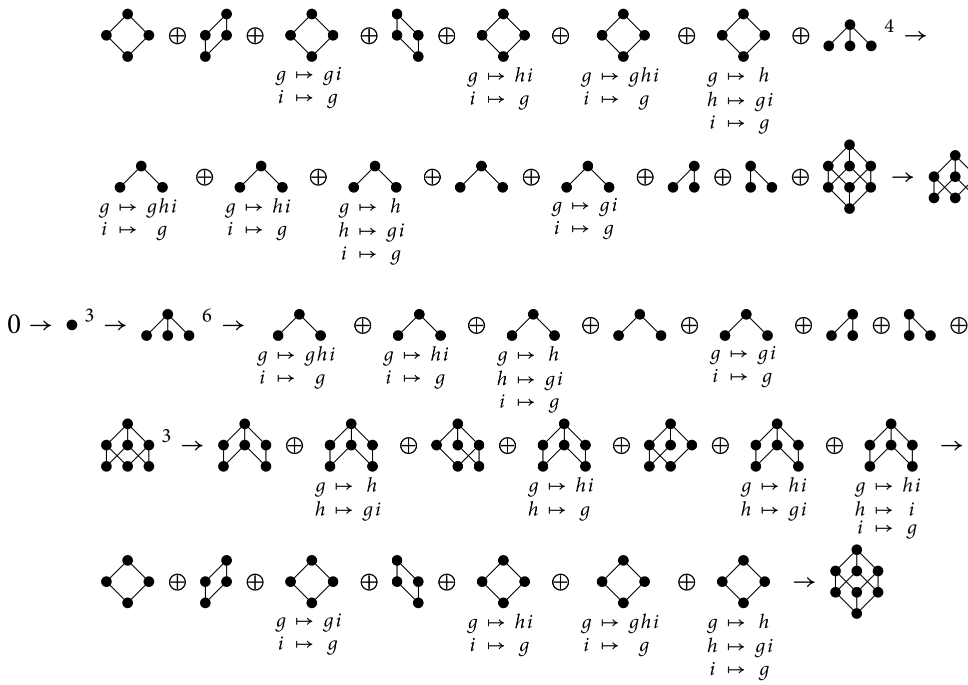


B.6 $C_2 \times C_2 \times C_2$

If $C_2 \times C_2 \times C_2 = \langle g, h, i \rangle$ then  represents the action of $g-1$;  represents the action of $h-1$; and  represents the action of $i-1$. Where there are modules that are equivalent up to a group automorphism we only include a resolution for one of these.

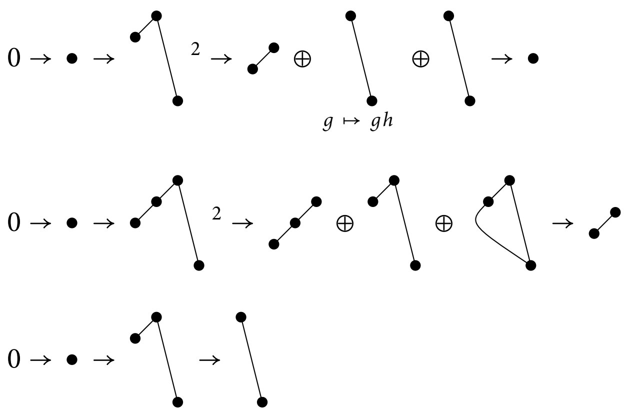


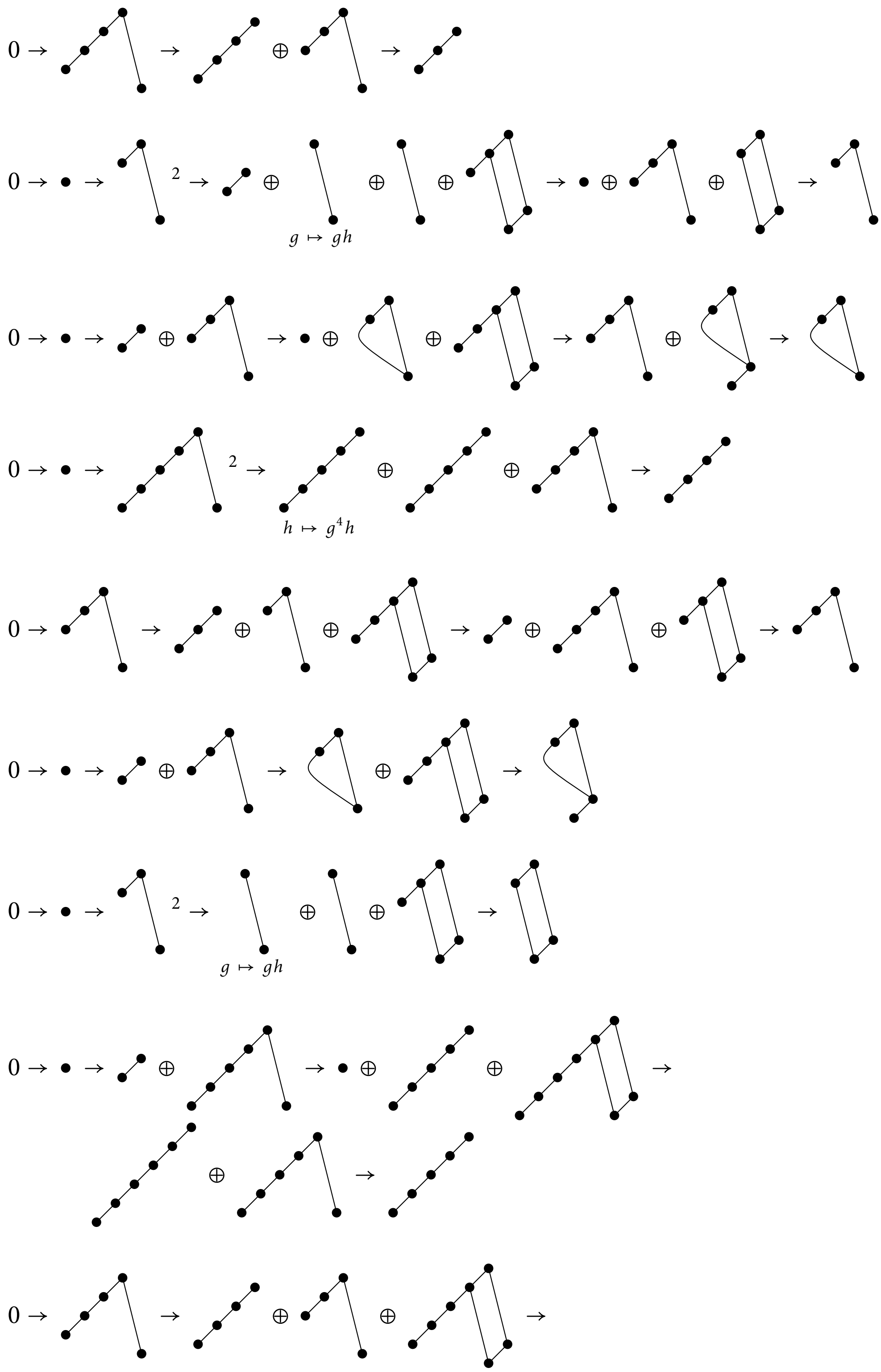

g ↦ hi
h ↦ g
g ↦ h
h ↦ gi
g ↦ hi
h ↦ gi
g ↦ hi
h ↦ i
i ↦ g
g ↦ hi
i ↦ g
g ↦ hi
h ↦ i
i ↦ g
g ↦ hi
h ↦ i
i ↦ g
g ↦ hi
h ↦ i
i ↦ g
g ↦ hi
h ↦ i
i ↦ g
g ↦ ghi
i ↦ g
g ↦ hi
i ↦ g
g ↦ h
h ↦ gi
i ↦ g
g ↦ gi
i ↦ g
g ↦ hi
h ↦ g
g ↦ h
h ↦ gi
g ↦ hi
h ↦ gi
g ↦ hi
h ↦ i
i ↦ g
g ↦ h
h ↦ gi
g ↦ hi
h ↦ g
g ↦ hi
h ↦ gi
g ↦ hi
h ↦ i
i ↦ g
g ↦ hi
i ↦ g
g ↦ ghi
i ↦ g
g ↦ hi
i ↦ g
g ↦ h
h ↦ gi
i ↦ g
g ↦ gi
i ↦ g
g ↦ h
h ↦ gi
g ↦ hi
h ↦ g
g ↦ hi
h ↦ gi
g ↦ hi
h ↦ i
i ↦ g

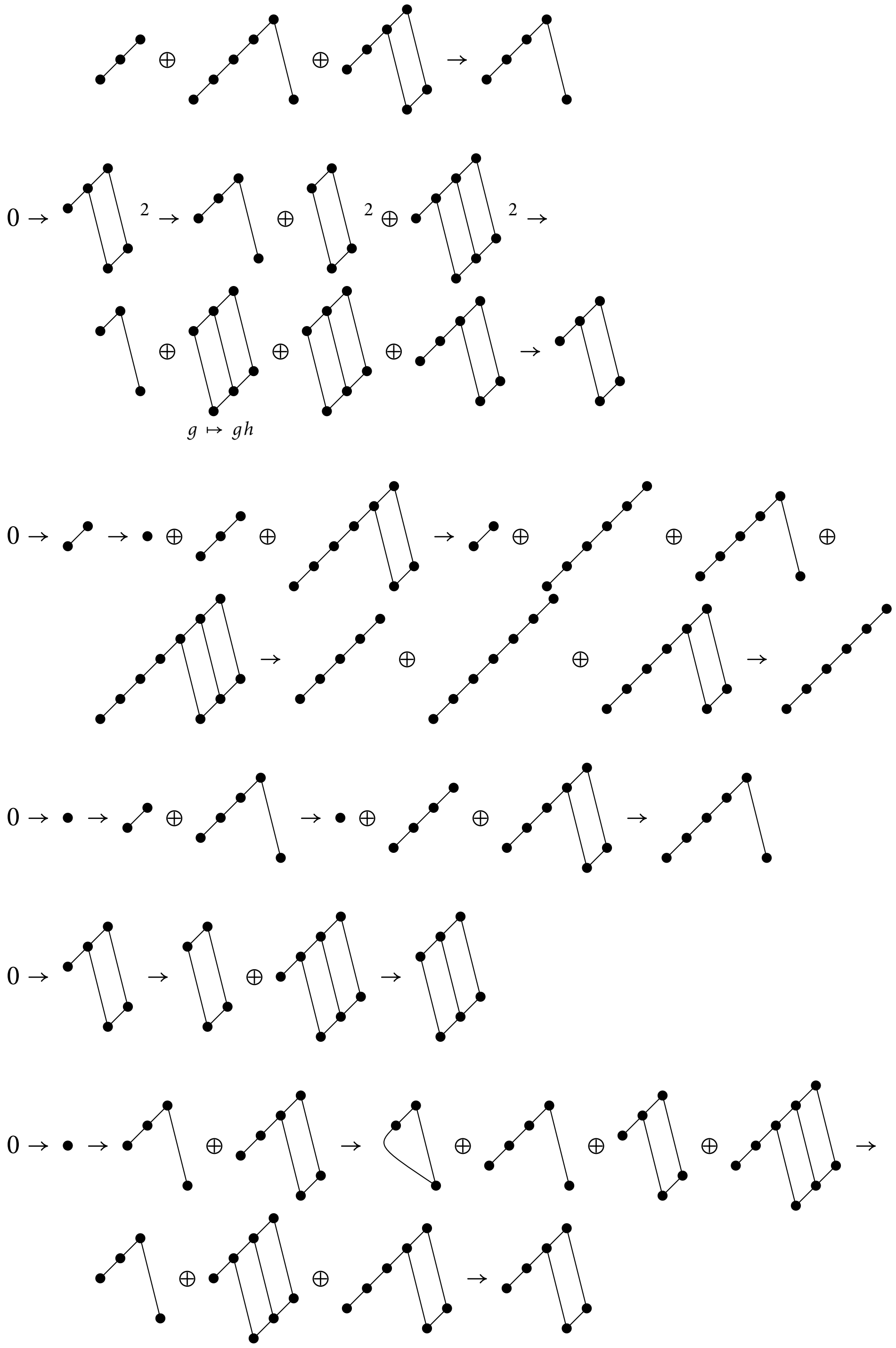


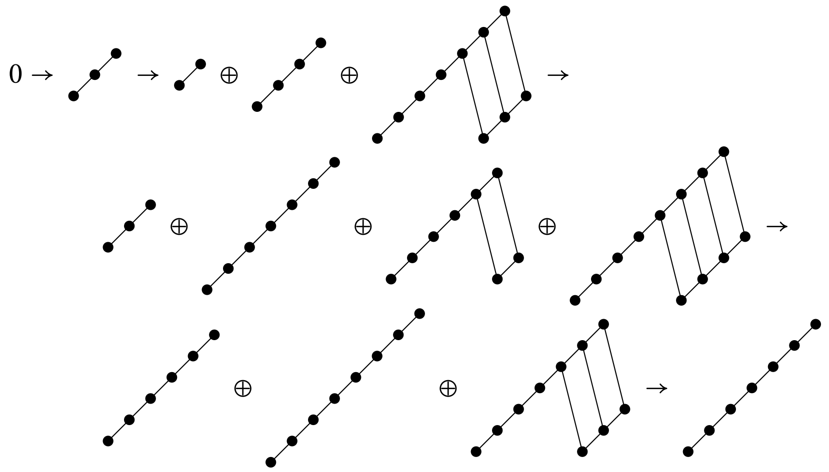
B.7 $C_8 \times C_2$

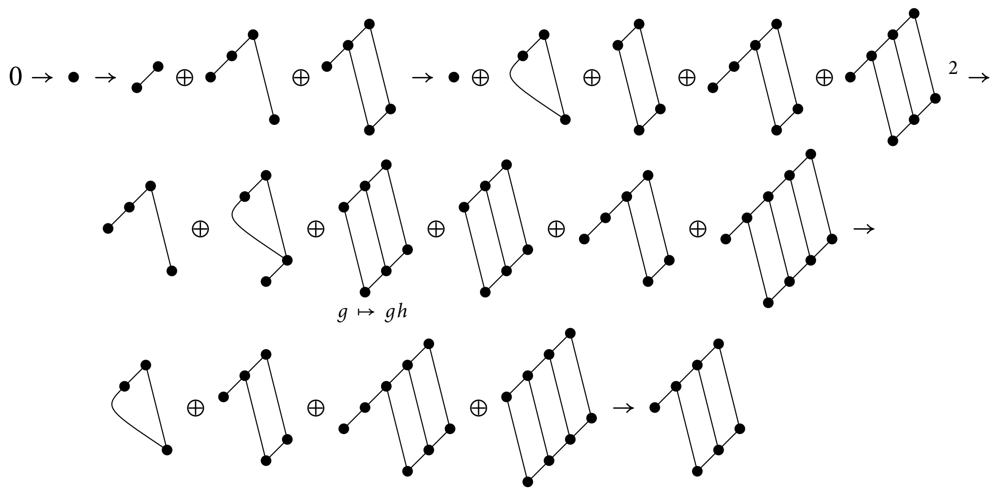
We use $\mathcal{F}_G$ and represent the vertices in the correct layer according to the standard filtration. If $C_8 \times C_2 = \langle g, h\rangle$ with $g^8 = h^2 = 1$ then  represents the action of $g - 1$ and  represents the action of $h - 1$.

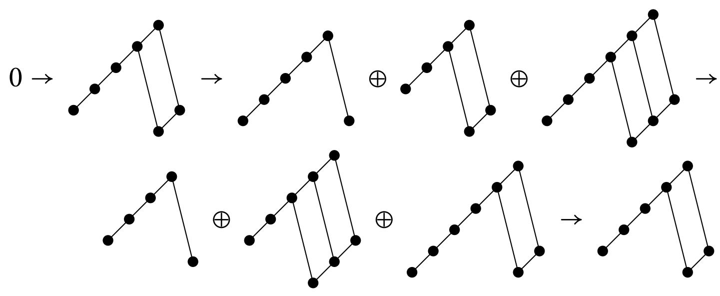


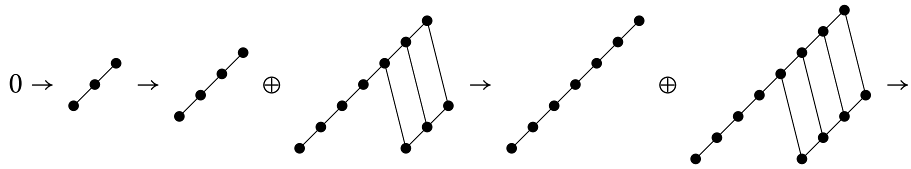

g ↦ gh
h ↦ g⁴h
g ↦ gh

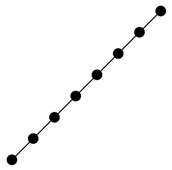

$g \mapsto gh$

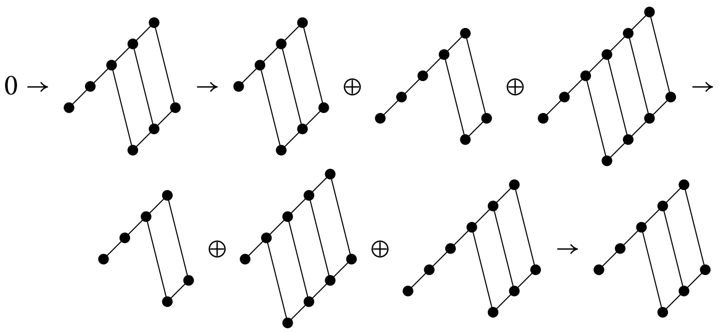


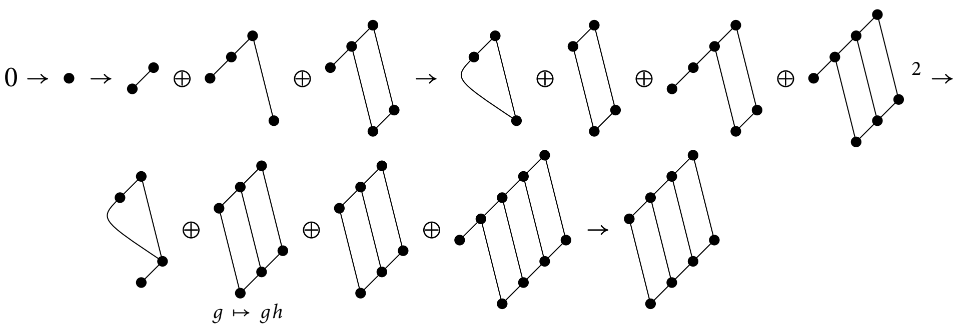

$g \mapsto gh$

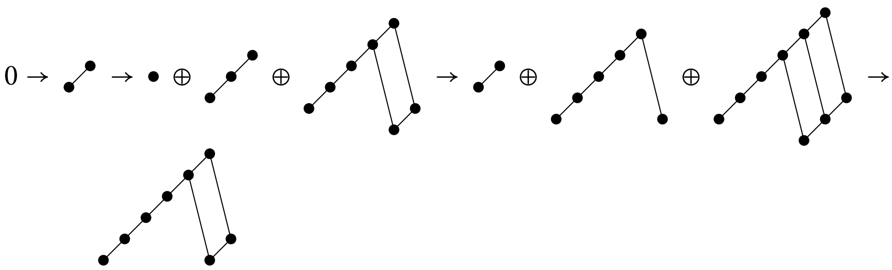


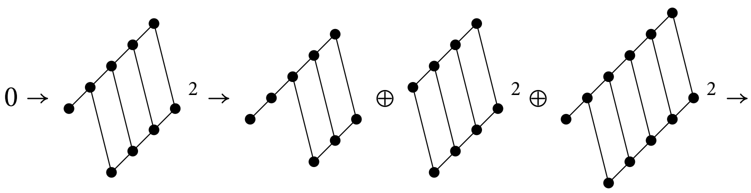


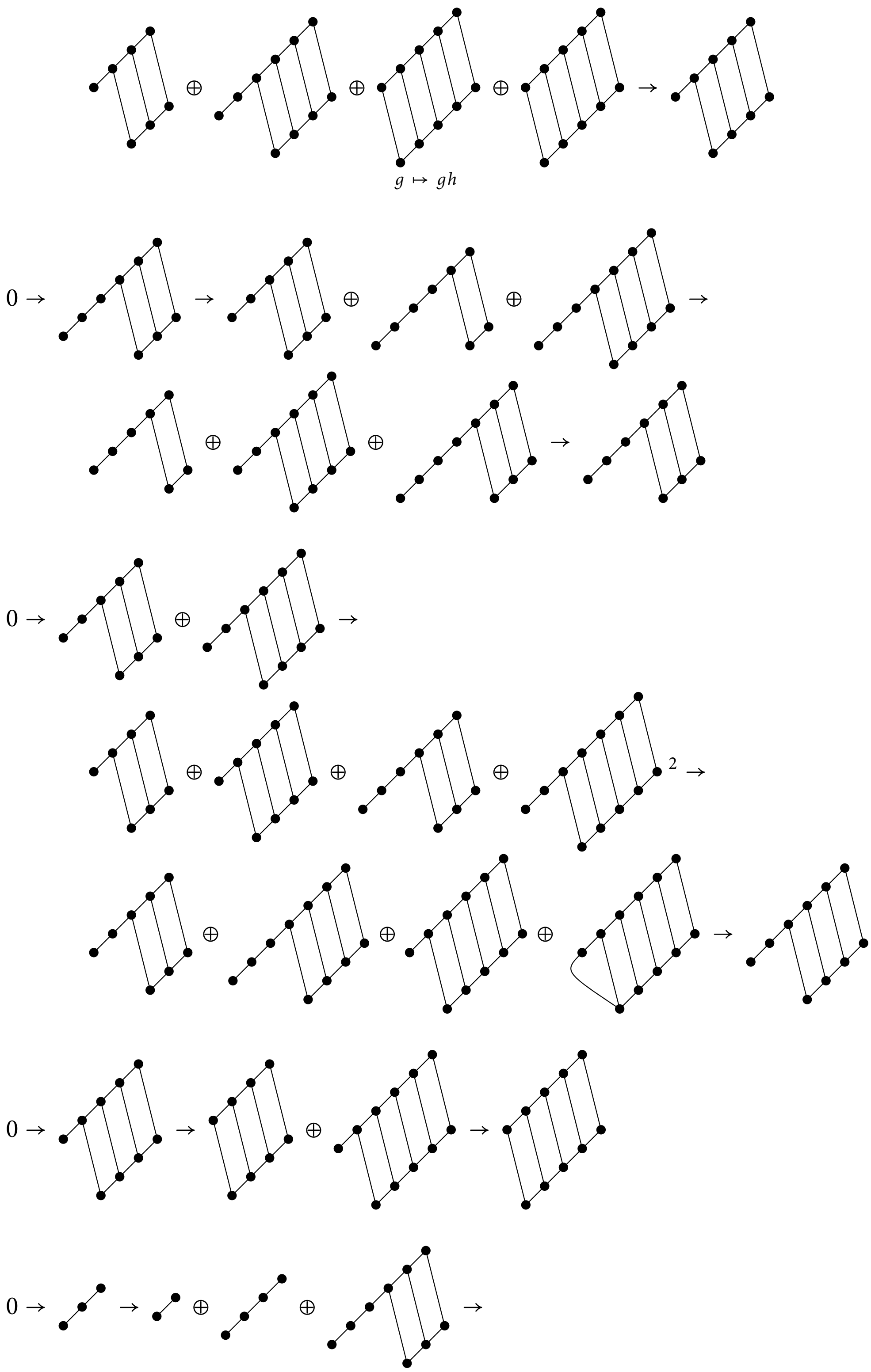


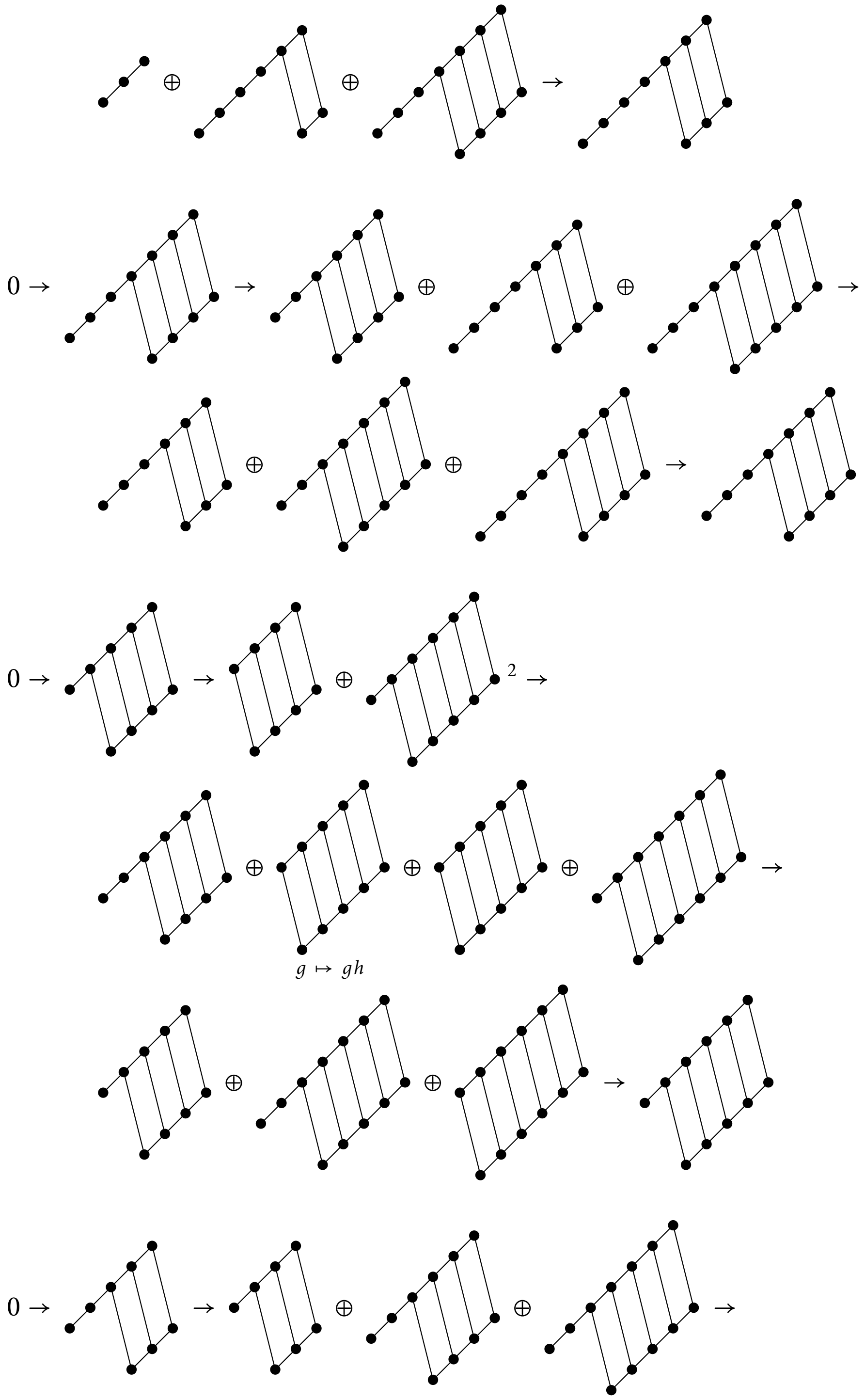


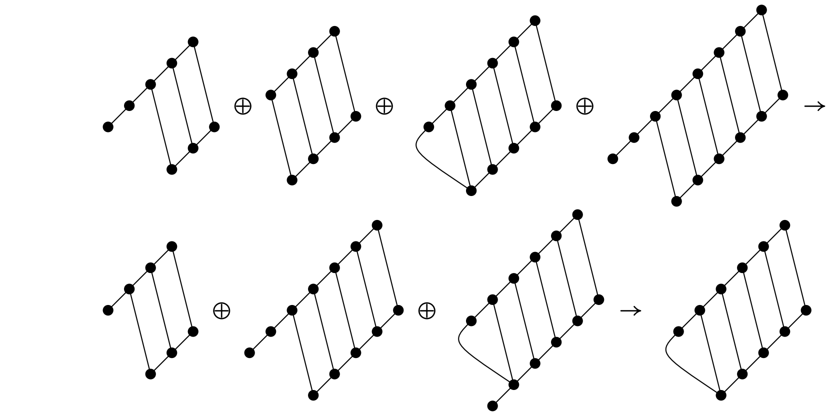

g ↦ gh

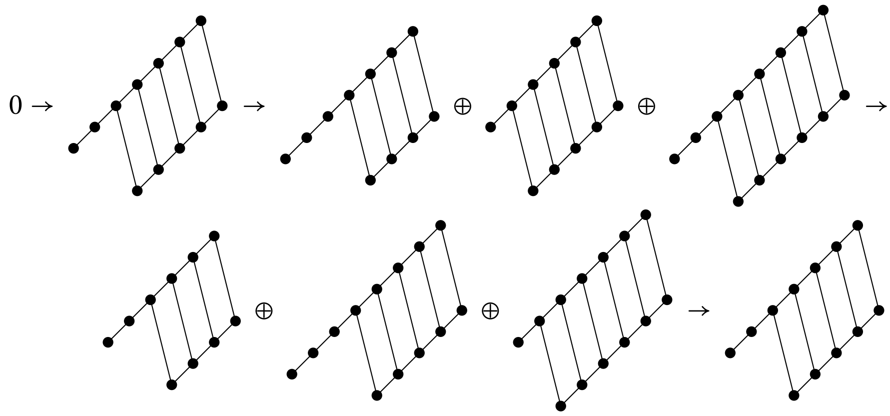


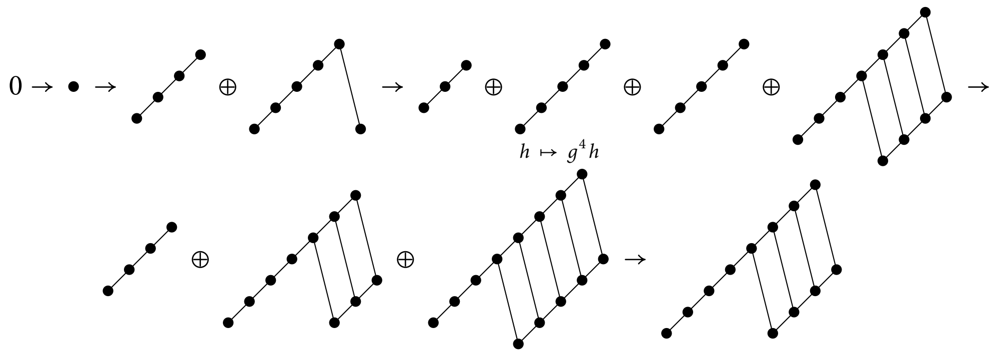


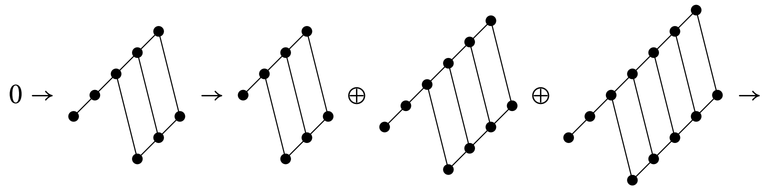

g ↦ gh
0 →
0 →
2
0 →
0 →

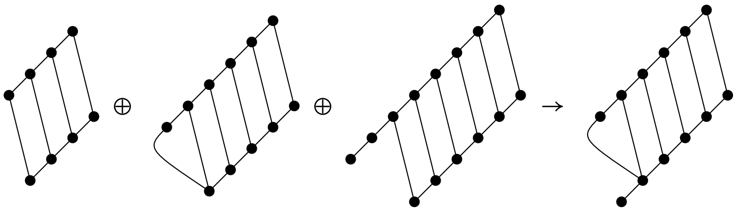

0 →
→
⊕
2
g ↦ gh

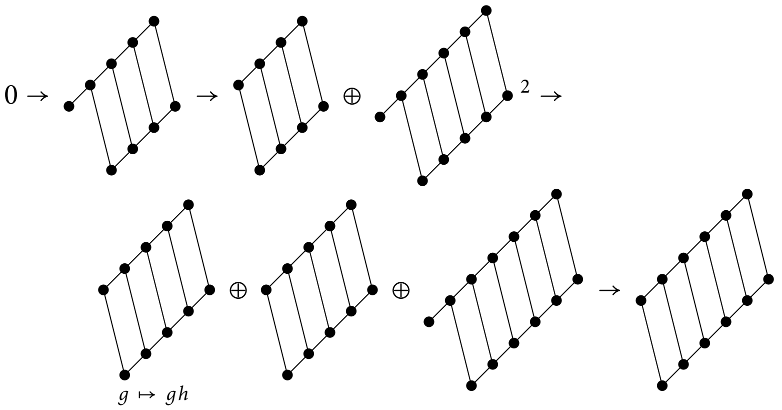


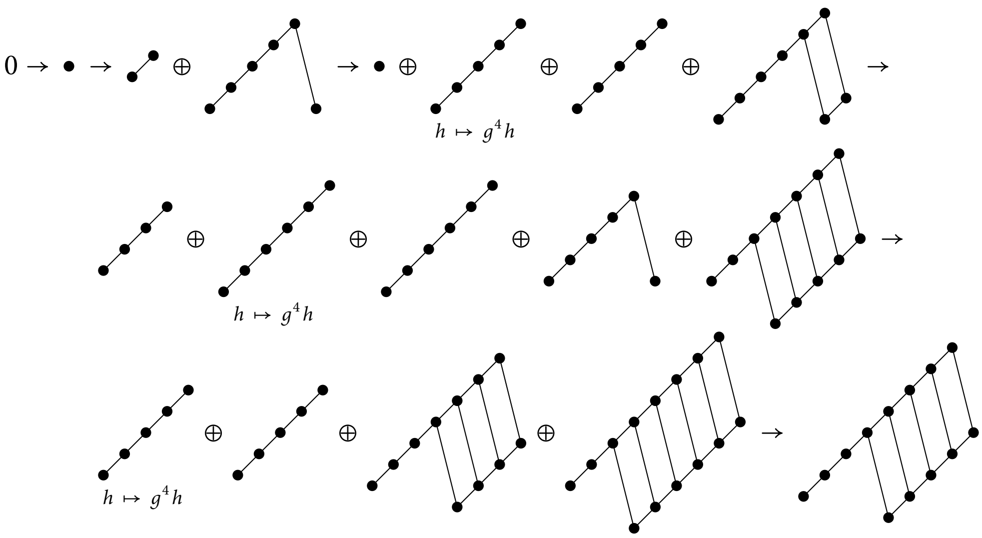


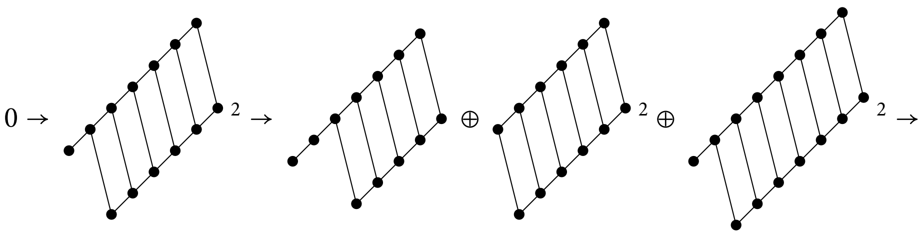

$h \mapsto g^4 h$

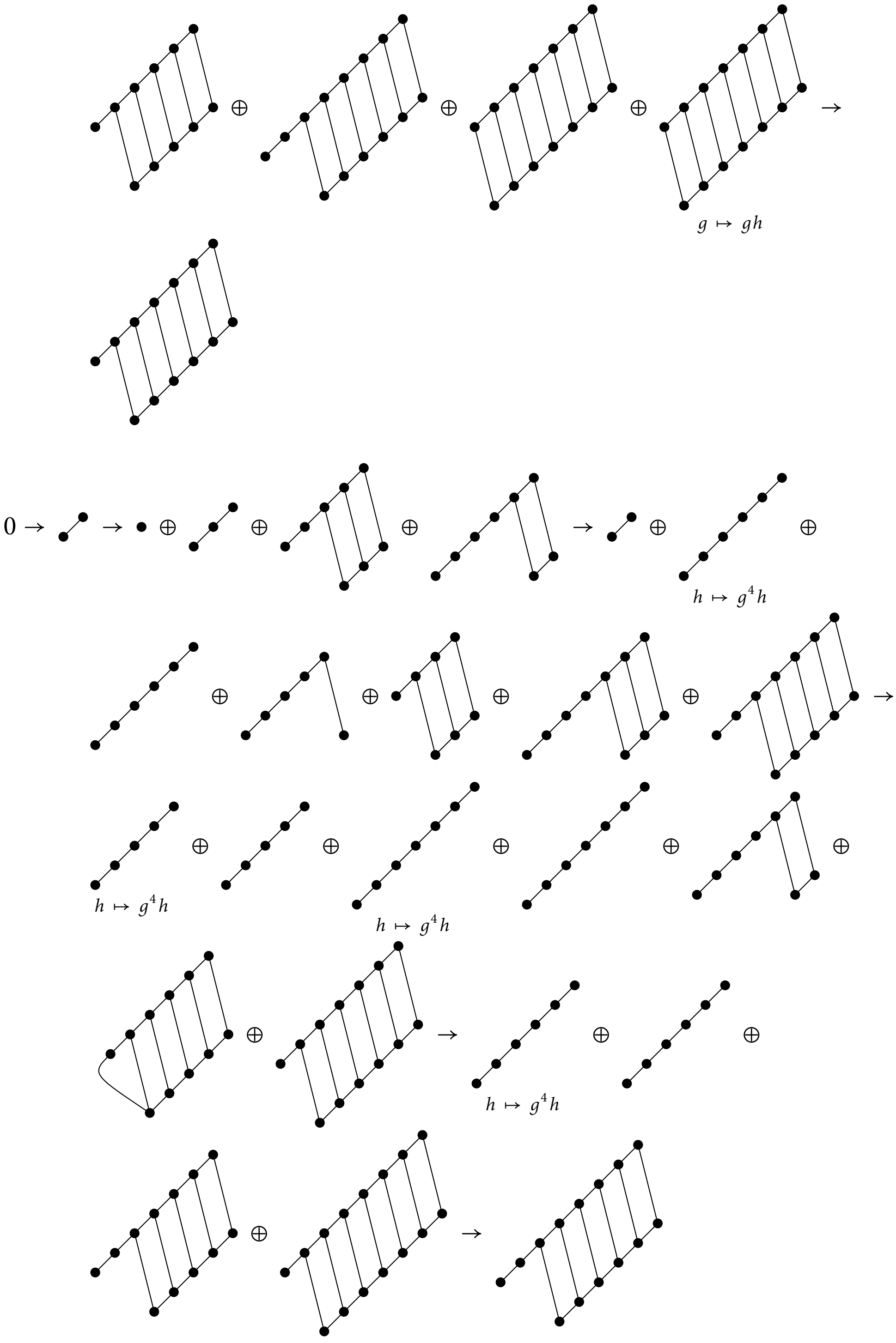


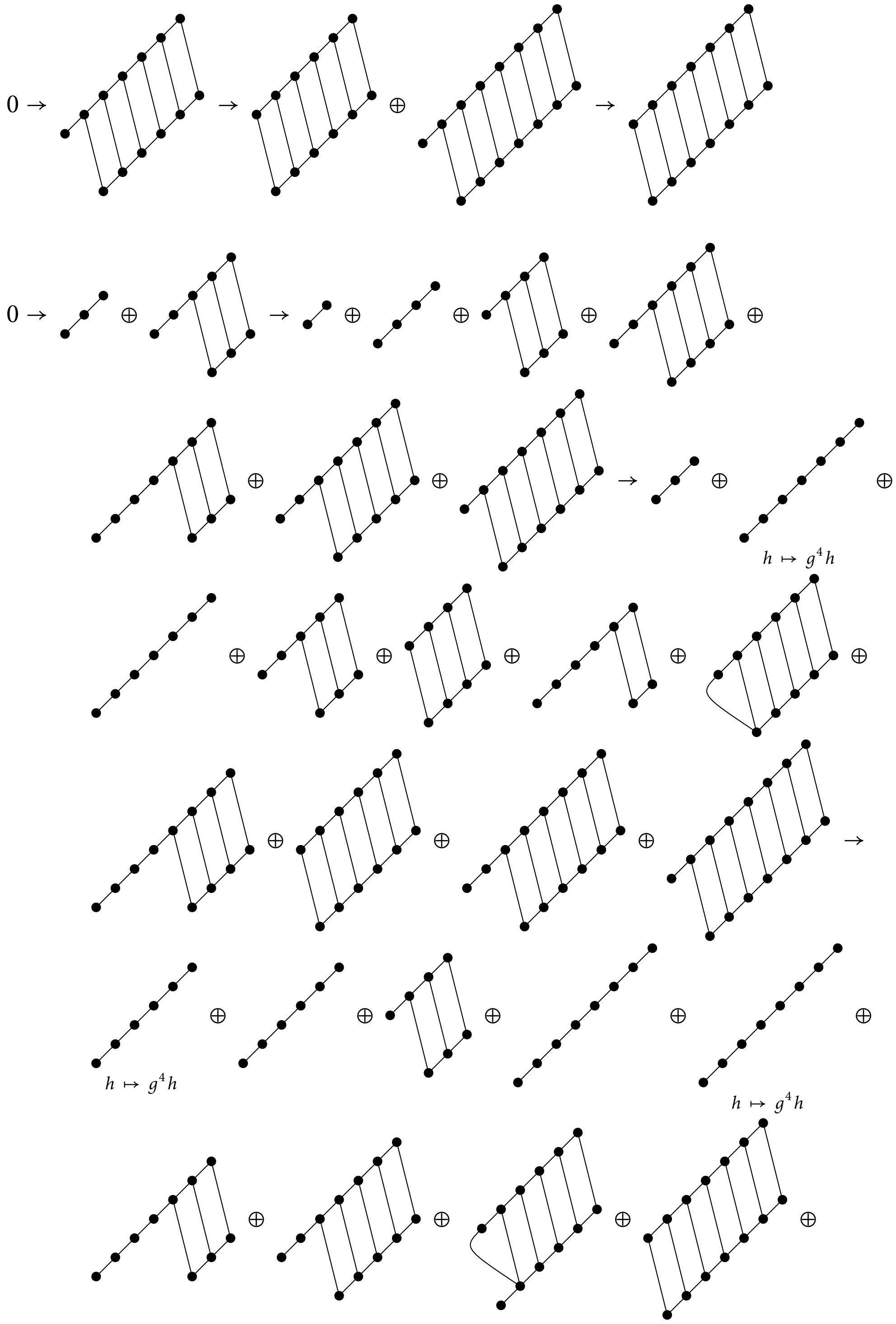


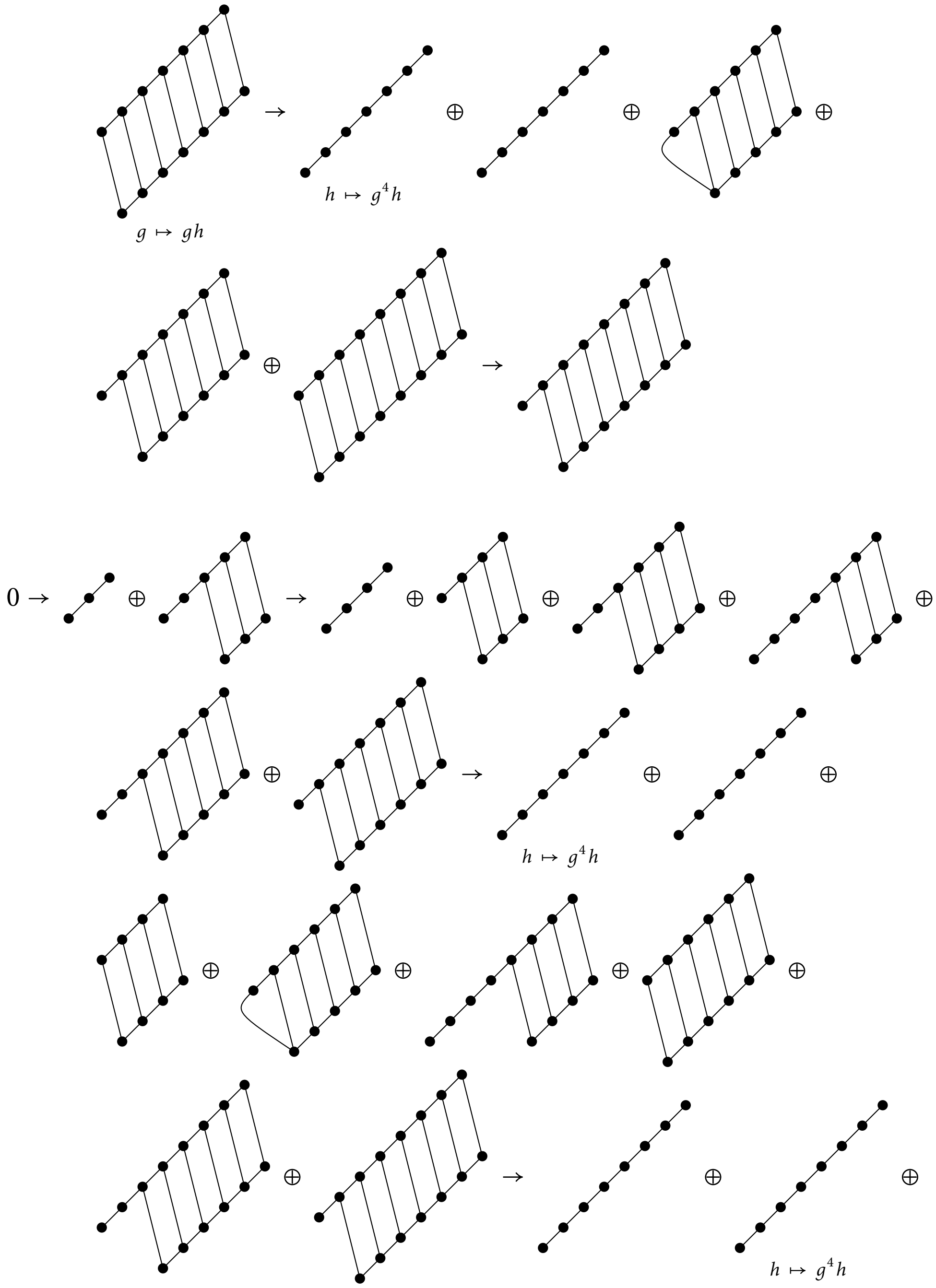

0 →
2
g ↦ gh

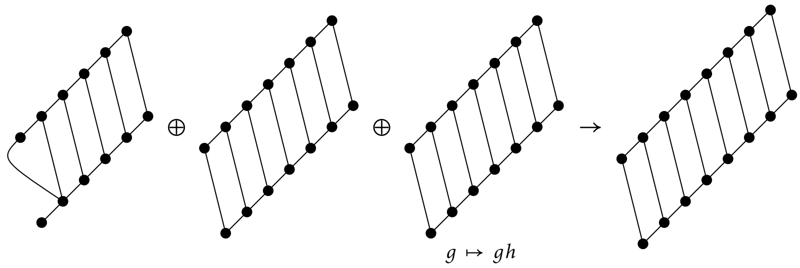

0 →
h ↦ g4h
h ↦ g4h
h ↦ g4h


0 →
2
2
2


$g \mapsto gh$
$h \mapsto g^4h$
$h \mapsto g^4h$
$h \mapsto g^4h$
$h \mapsto g^4h$

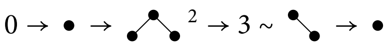

0 →
0 →
$h \mapsto g^4 h$
$h \mapsto g^4 h$
$h \mapsto g^4 h$

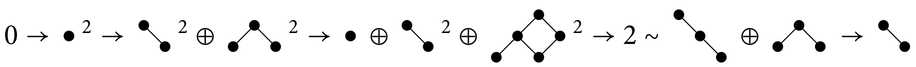

h ↦ g⁴h
g ↦ gh
0 →
h ↦ g⁴h
h ↦ g⁴h

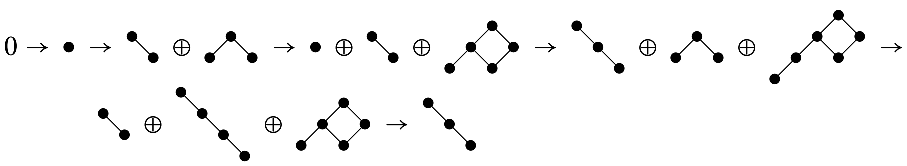


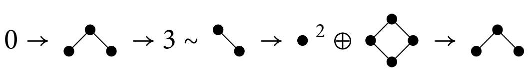
B.8 $C_4 \times C_4$

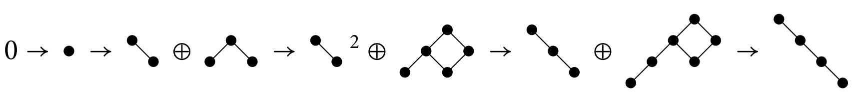
If $C_4 \times C_4 = \langle g, h \rangle$ then  represents the action of $g-1$ and  represents the action of $h-1$. In many cases several modules that are equivalent up to automorphism of the group appear in the resolutions, for succinctness we will use the notation $n \sim M$ to mean the direct sum of n non-isomorphic modules that are equivalent to M up to group automorphism.

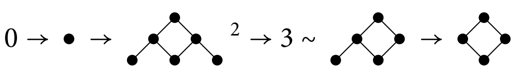


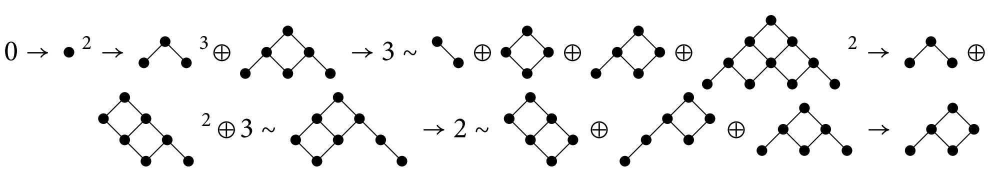


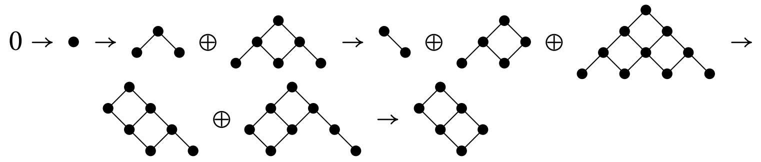


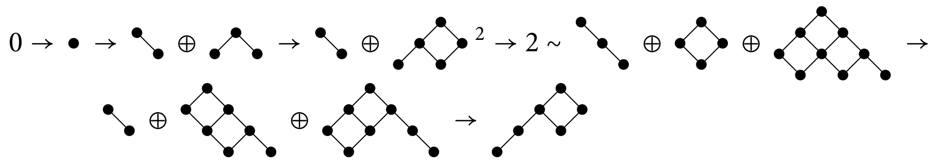


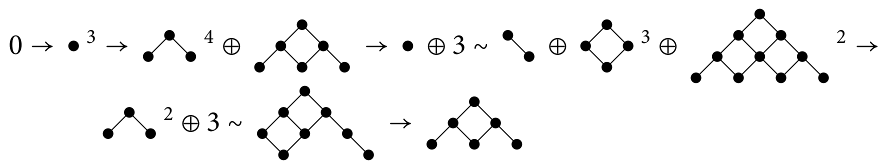


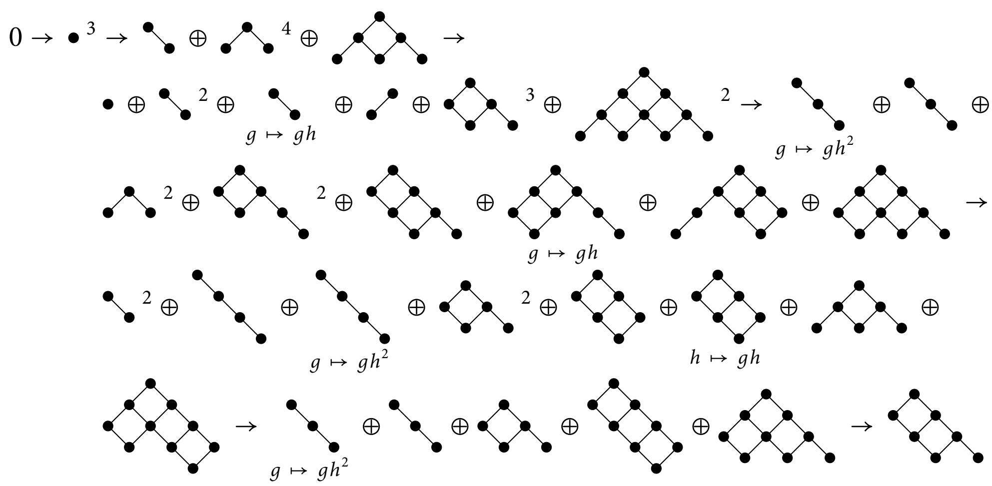


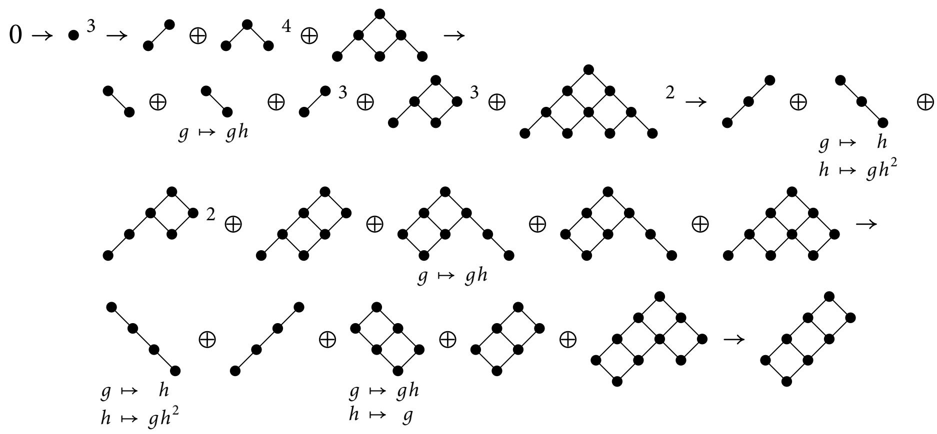


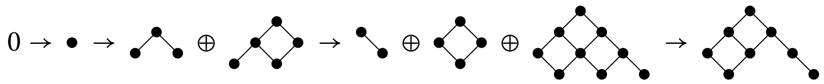


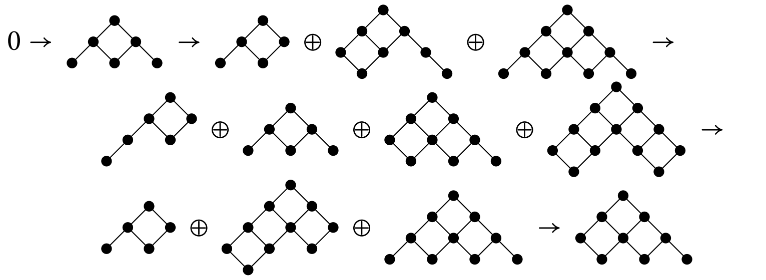


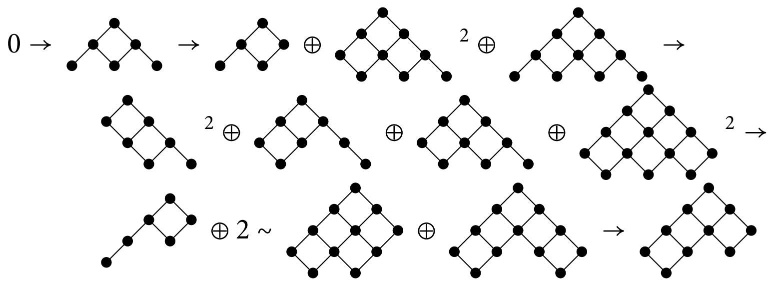


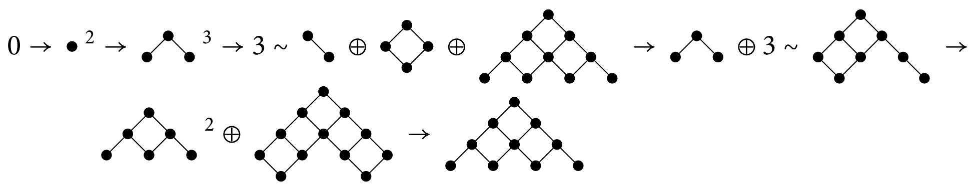

g ↦ gh
g ↦ gh²
g ↦ gh
g ↦ gh²
h ↦ gh
g ↦ gh²

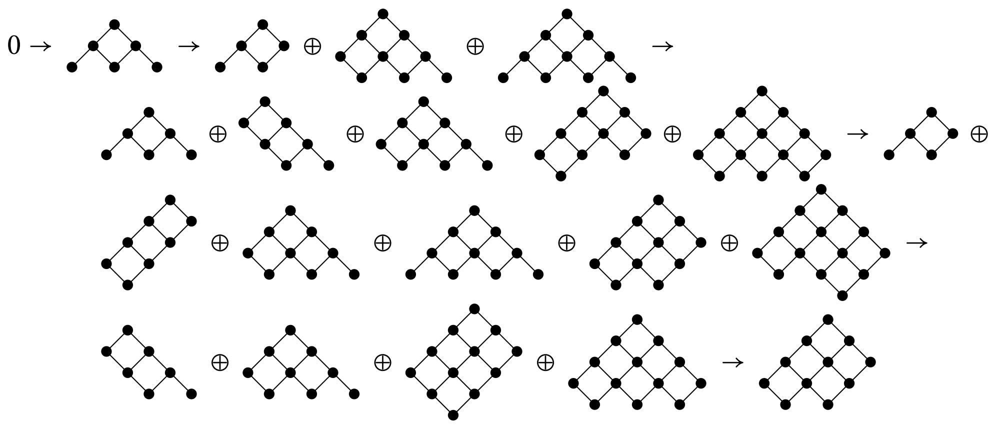

g ↦ gh
g ↦ h
h ↦ gh²
g ↦ gh
g ↦ h
h ↦ gh²
g ↦ gh
h ↦ g

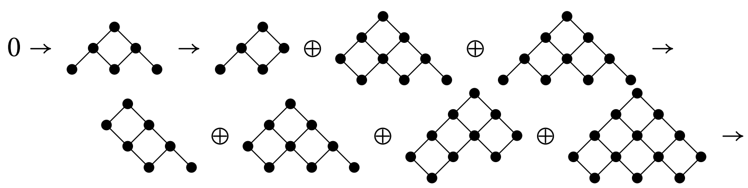


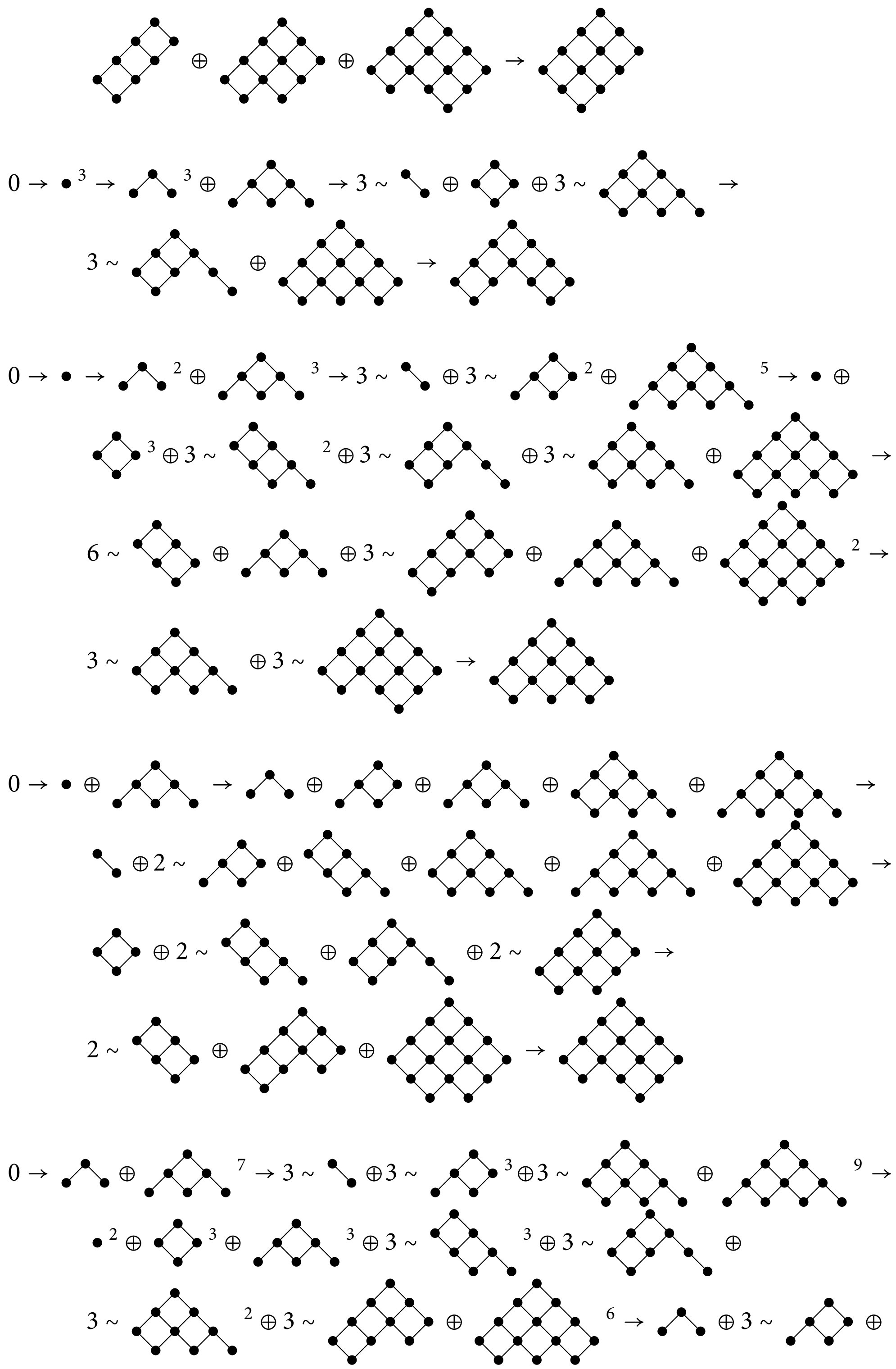


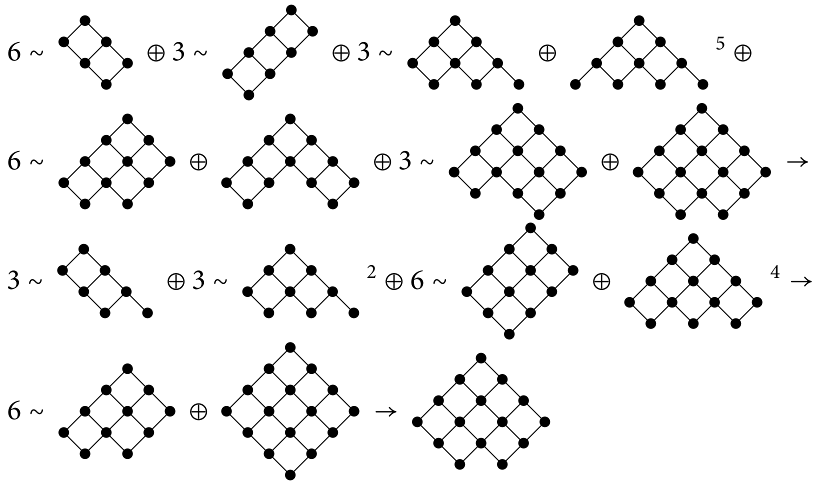


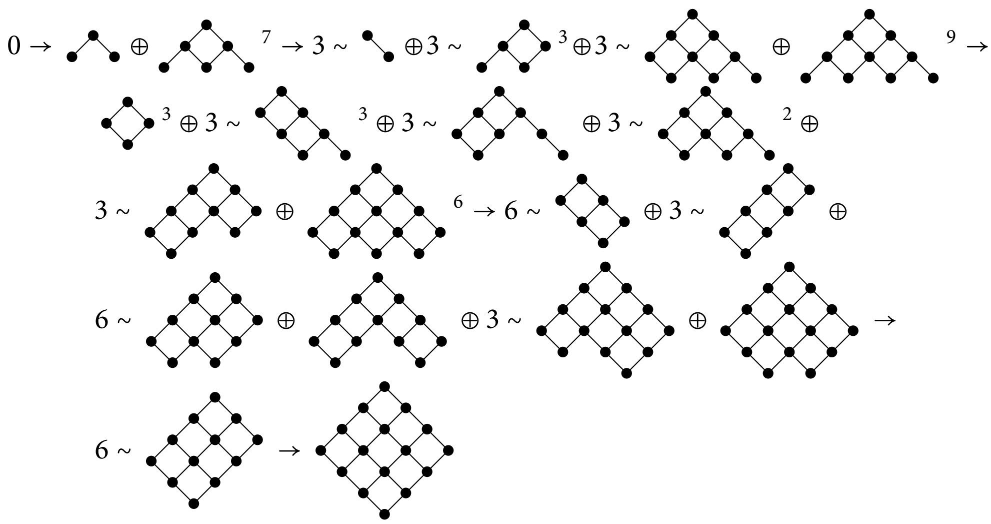




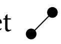


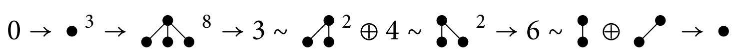


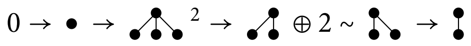


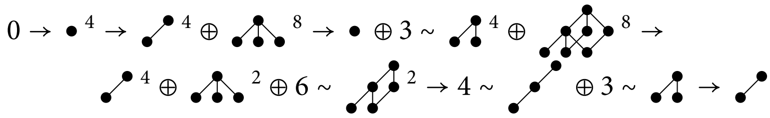


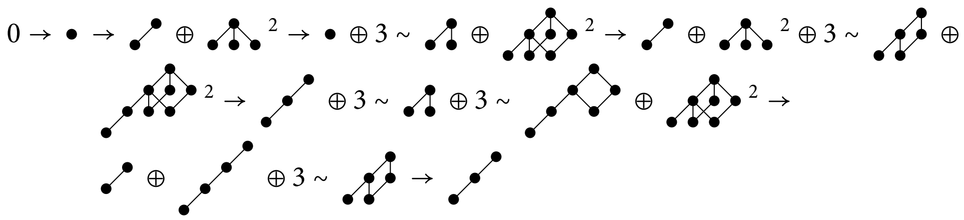
B.9 $C_4 \times C_2 \times C_2$

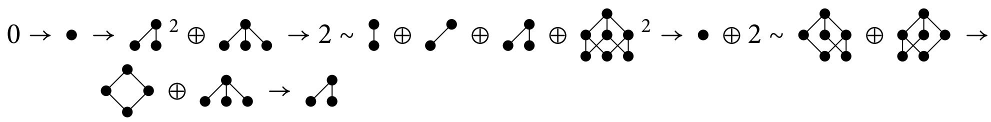
Here we use the class of modules $\mathcal{R}_G$, which is closed under quotients by radical powers, in order to calculate a bound on the representation dimension. For $C_4 \times C_2 \times C_2 = \langle g, h, i \rangle$ where $g^4 = h^2 = i^2$, we let  represent the action of $g - 1$;  represent the action of $h - 1$; and  represent the action of $i - 1$. As in the previous example we use the notation $n \sim M$ to represent the direct sum of n non-isomorphic modules that are equivalent up to automorphism of the group.

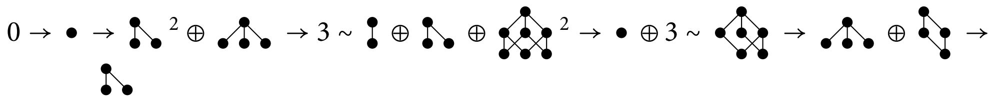


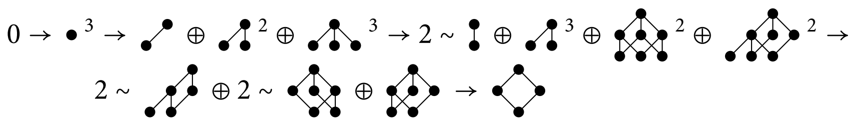


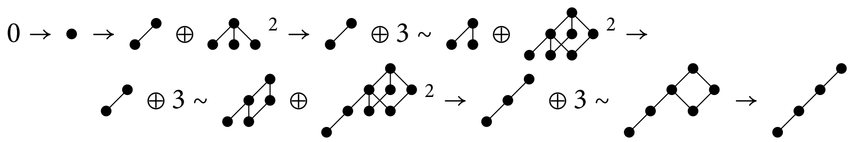


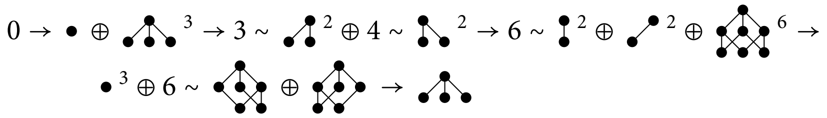


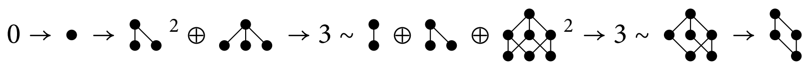


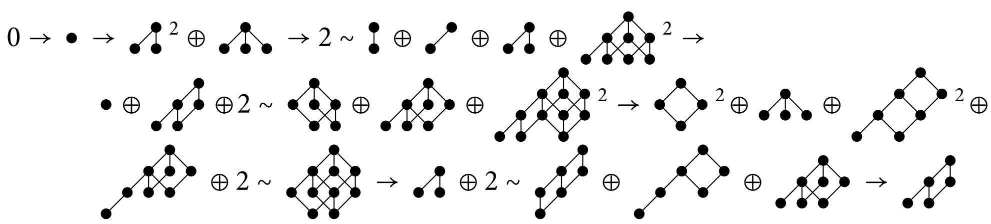


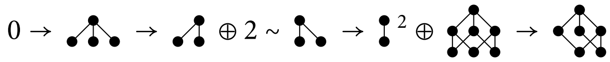


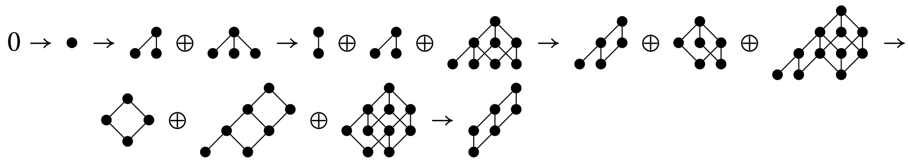


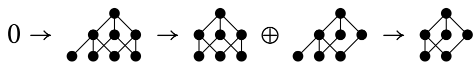


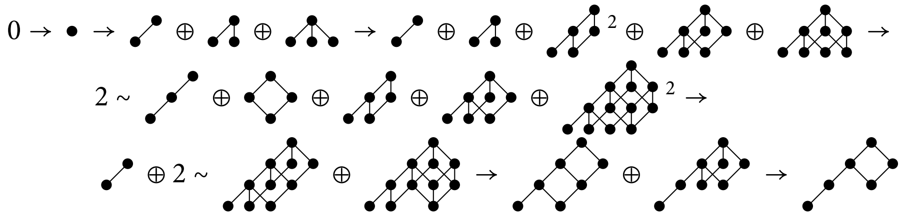


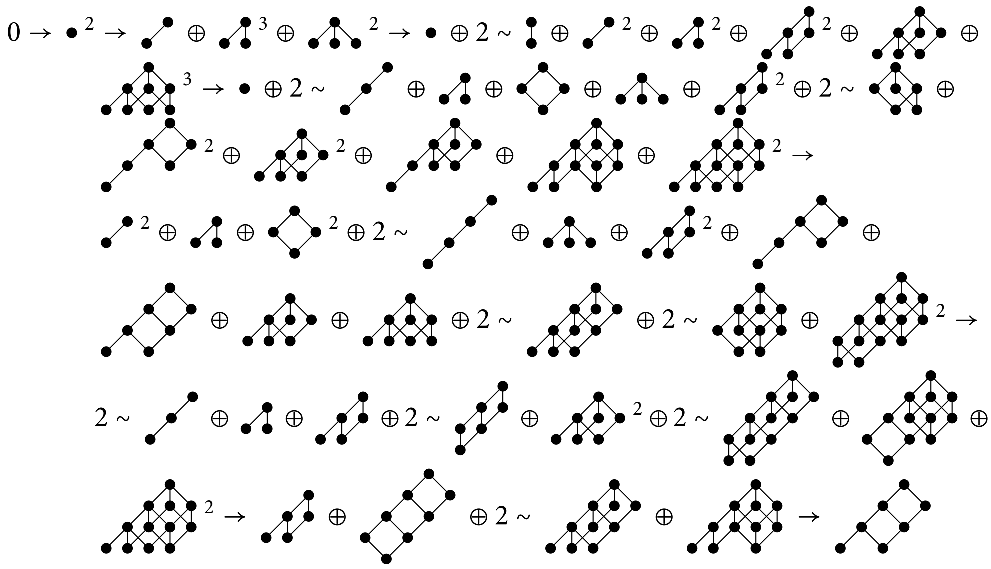


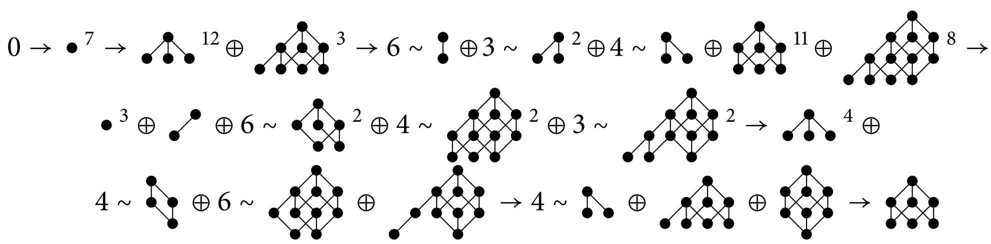


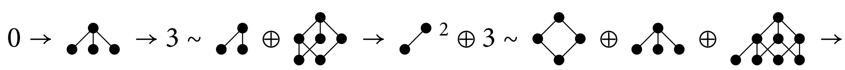


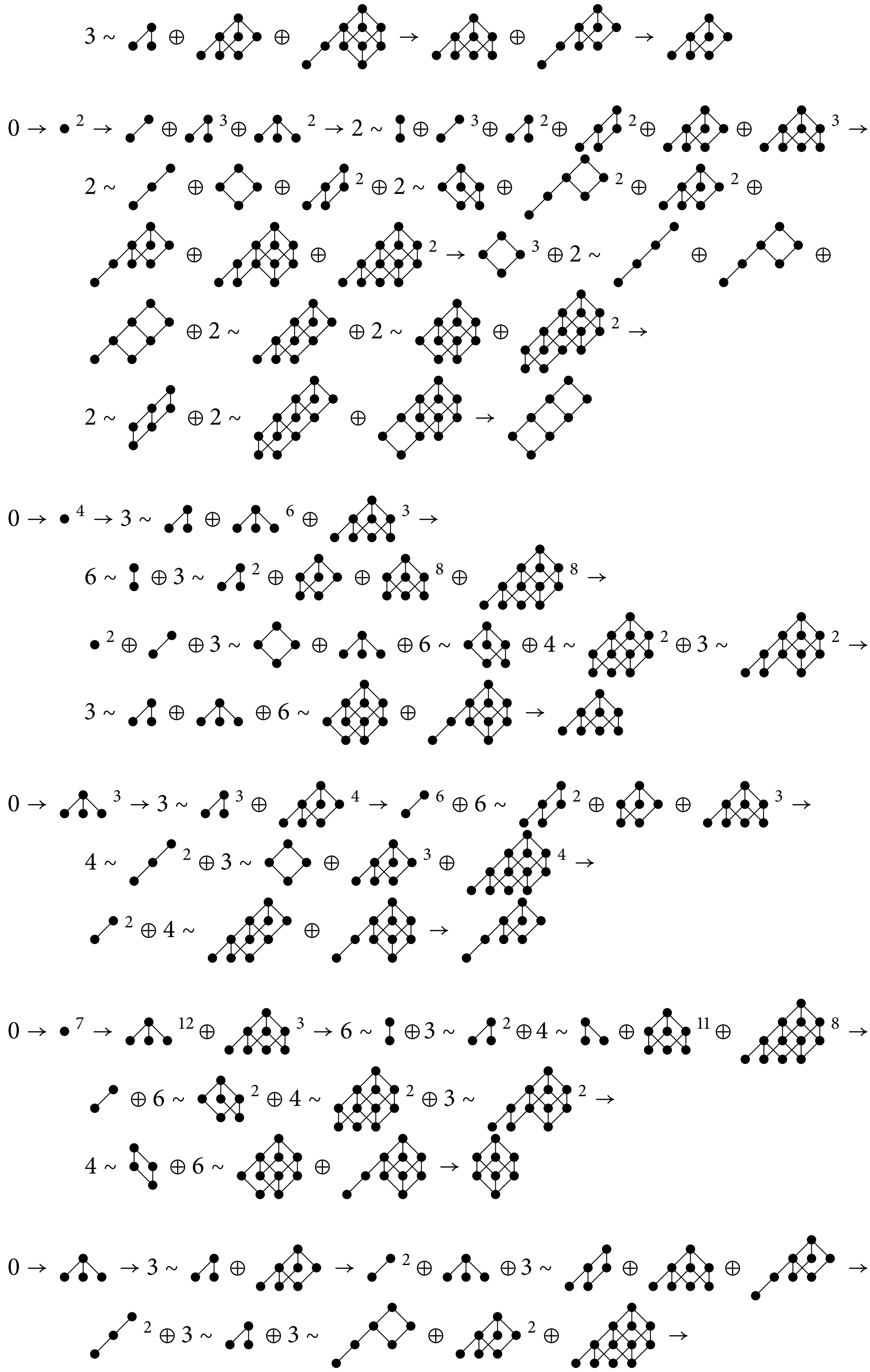


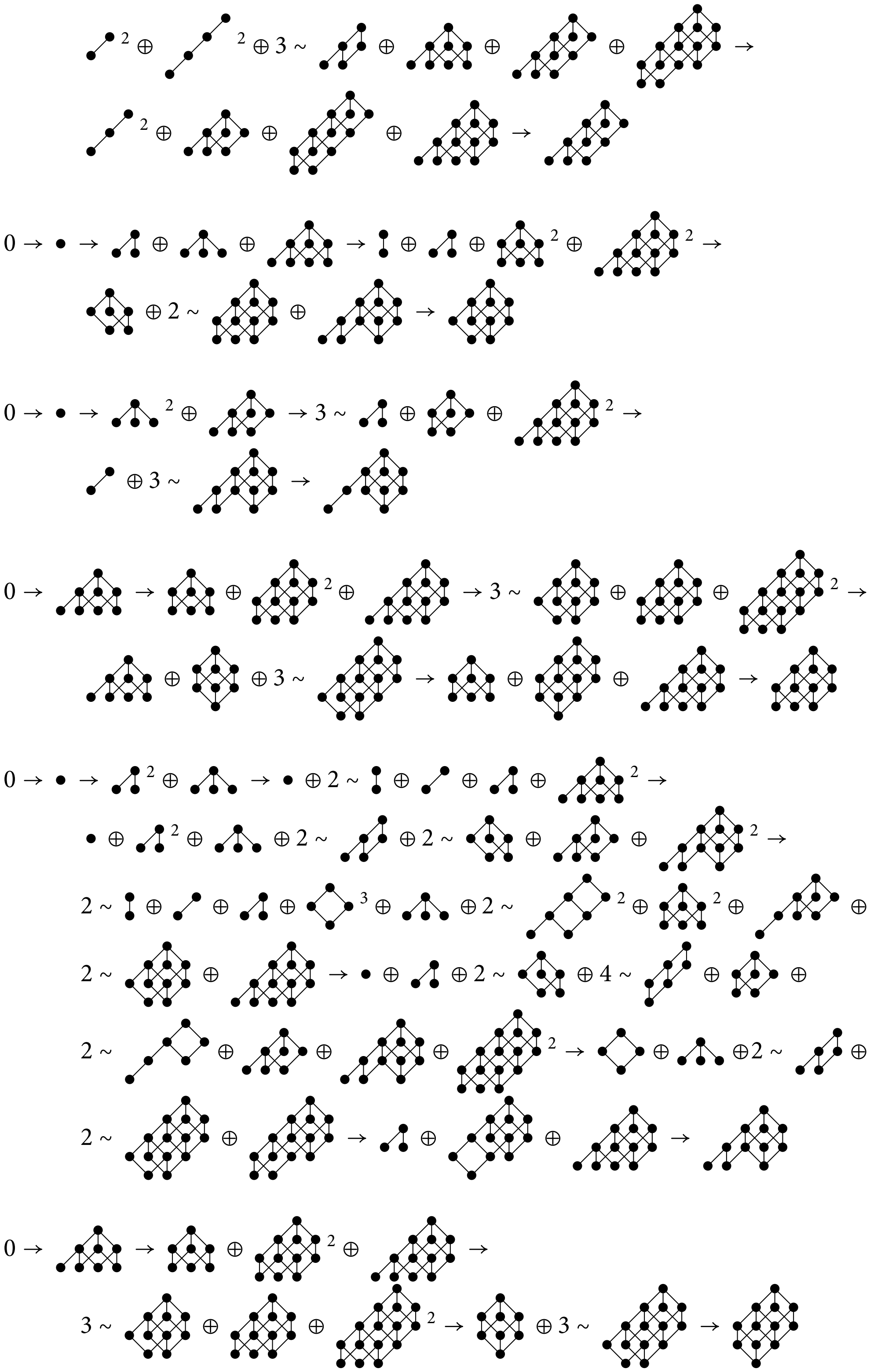


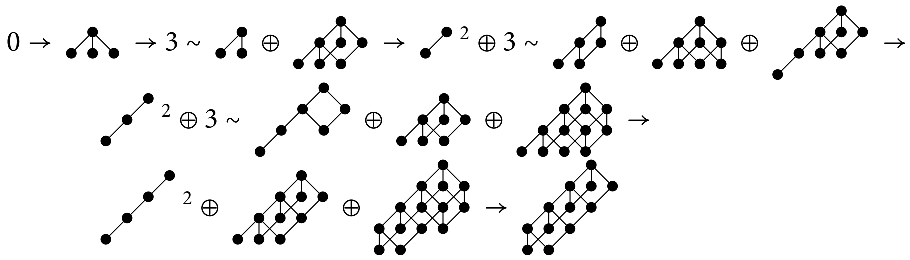


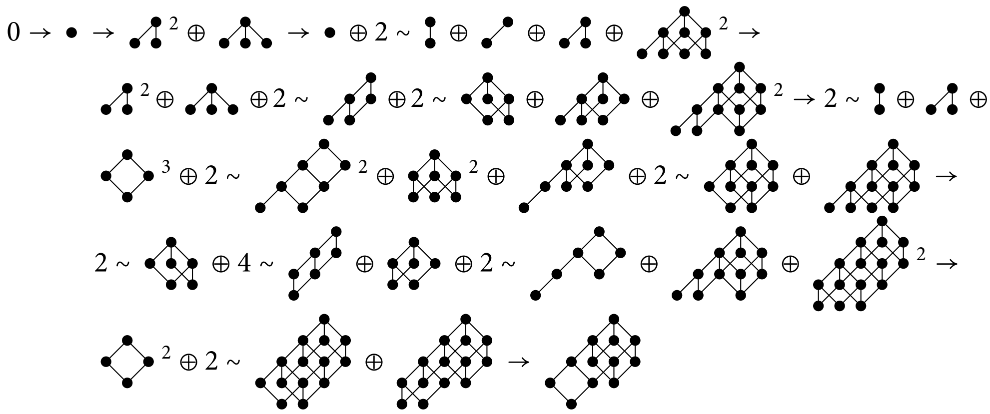


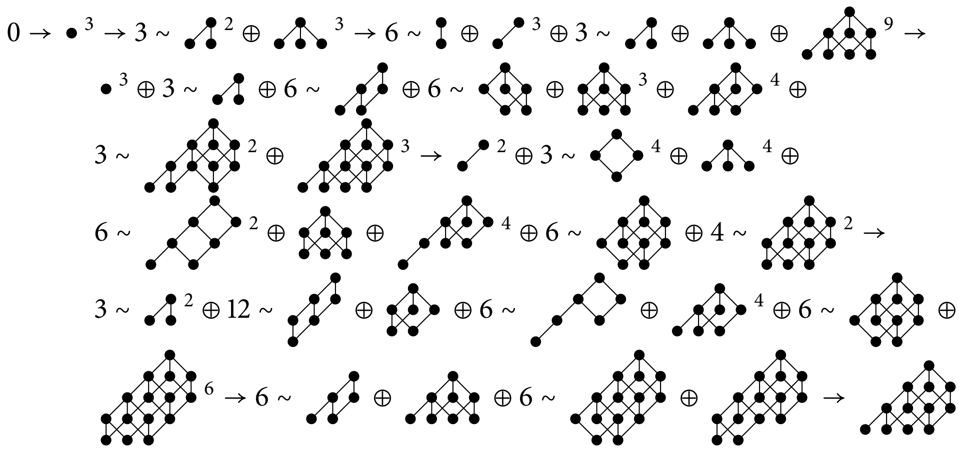


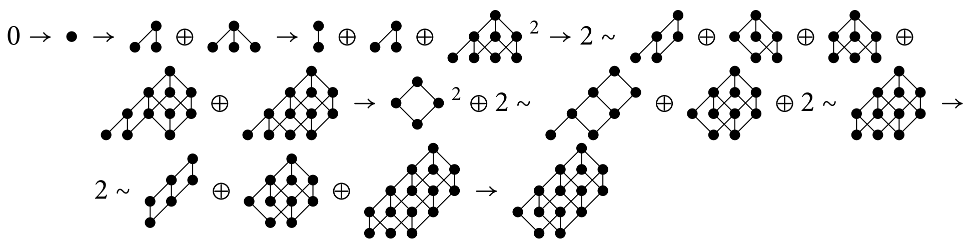


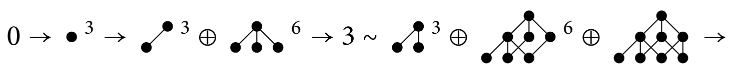


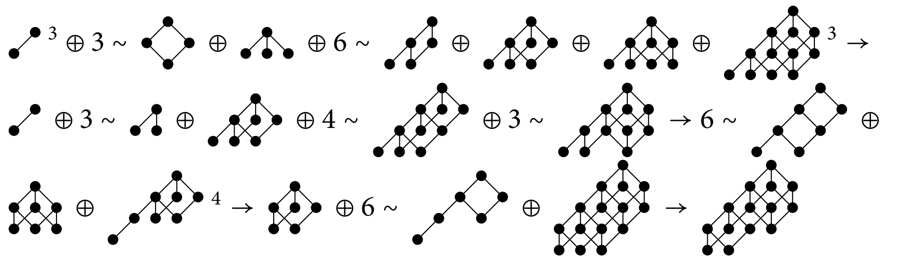


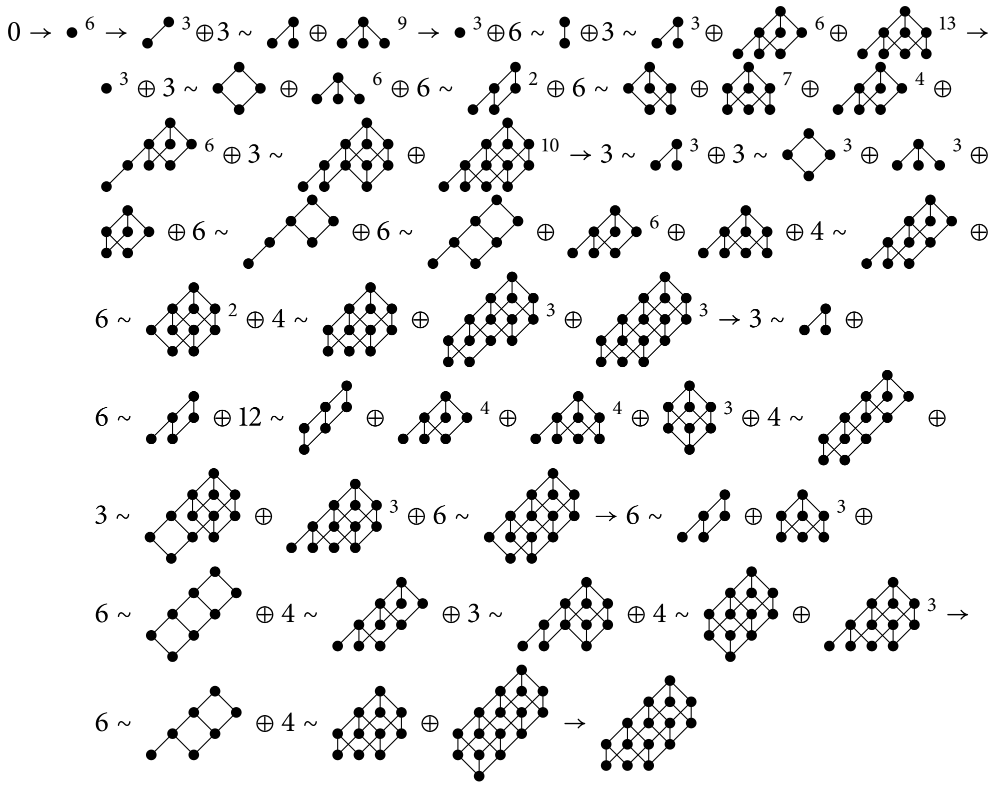


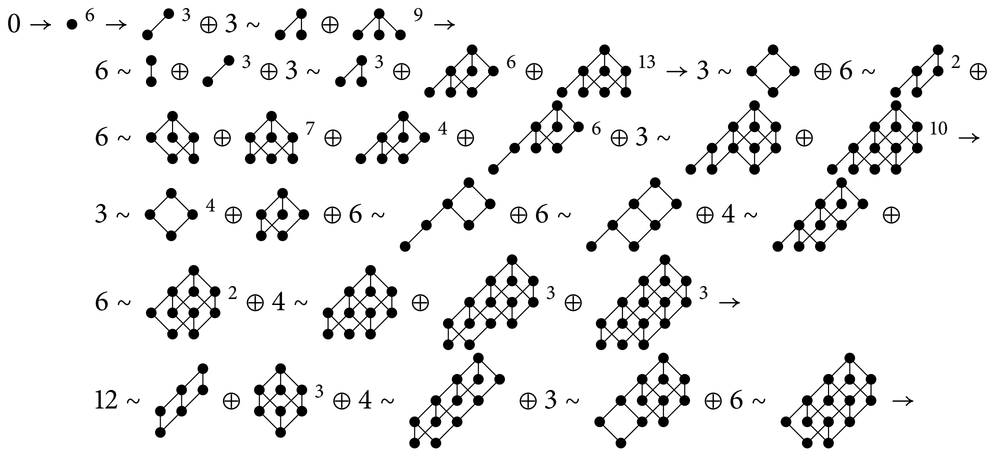


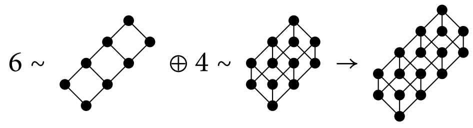




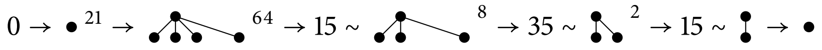


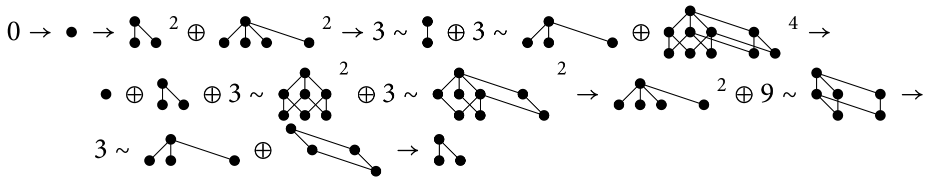


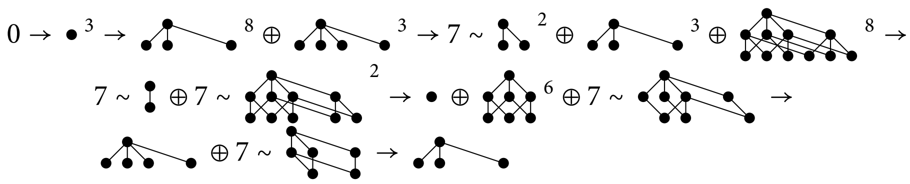


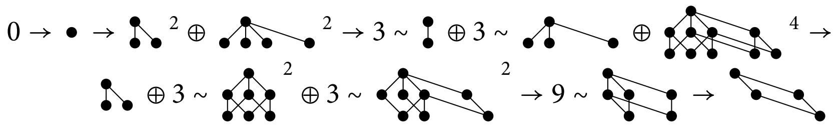


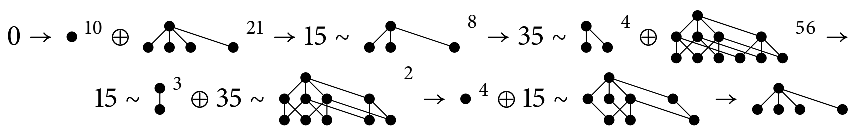
B.10 $C_2 \times C_2 \times C_2 \times C_2$

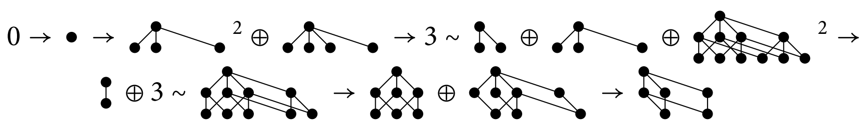


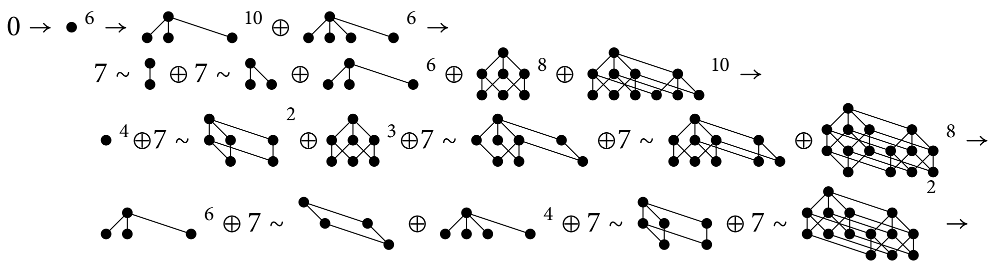


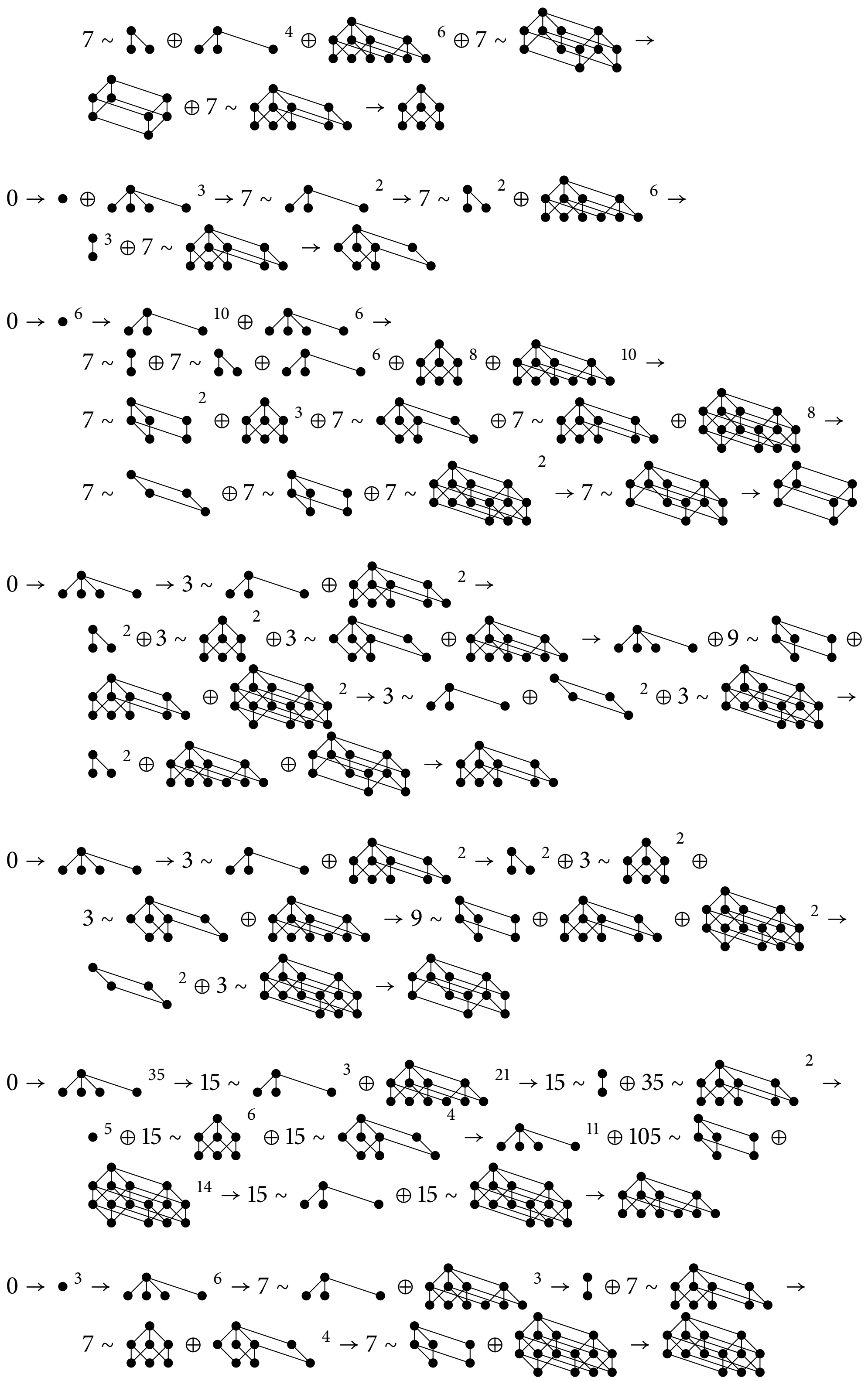


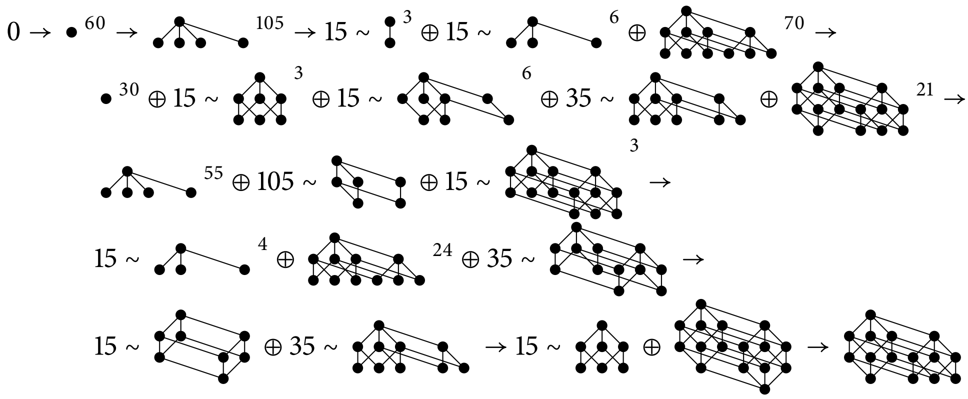


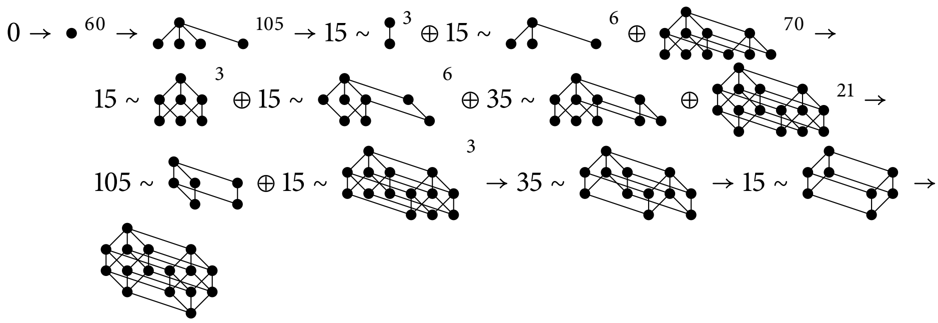




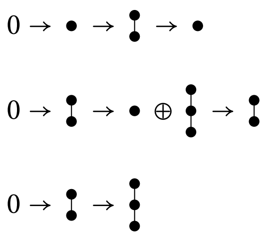




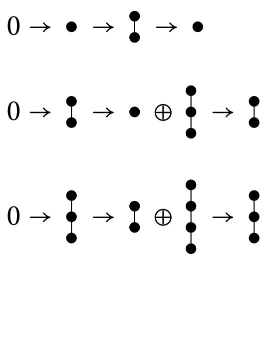


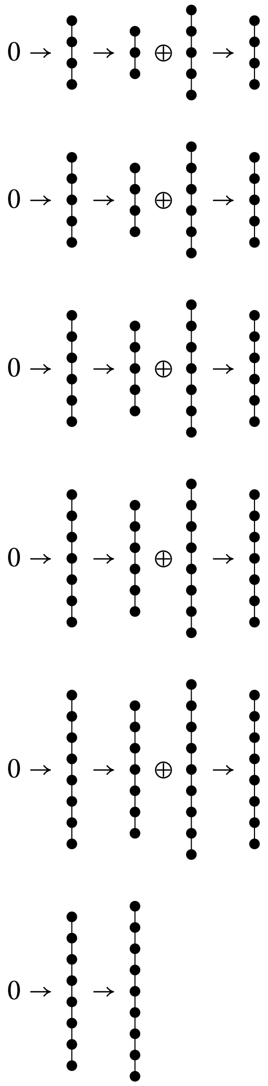


B.11 C_3

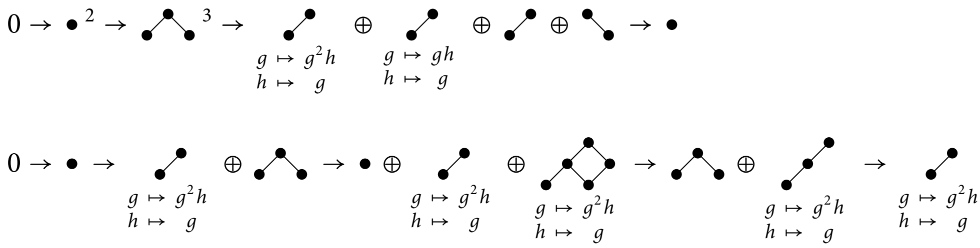


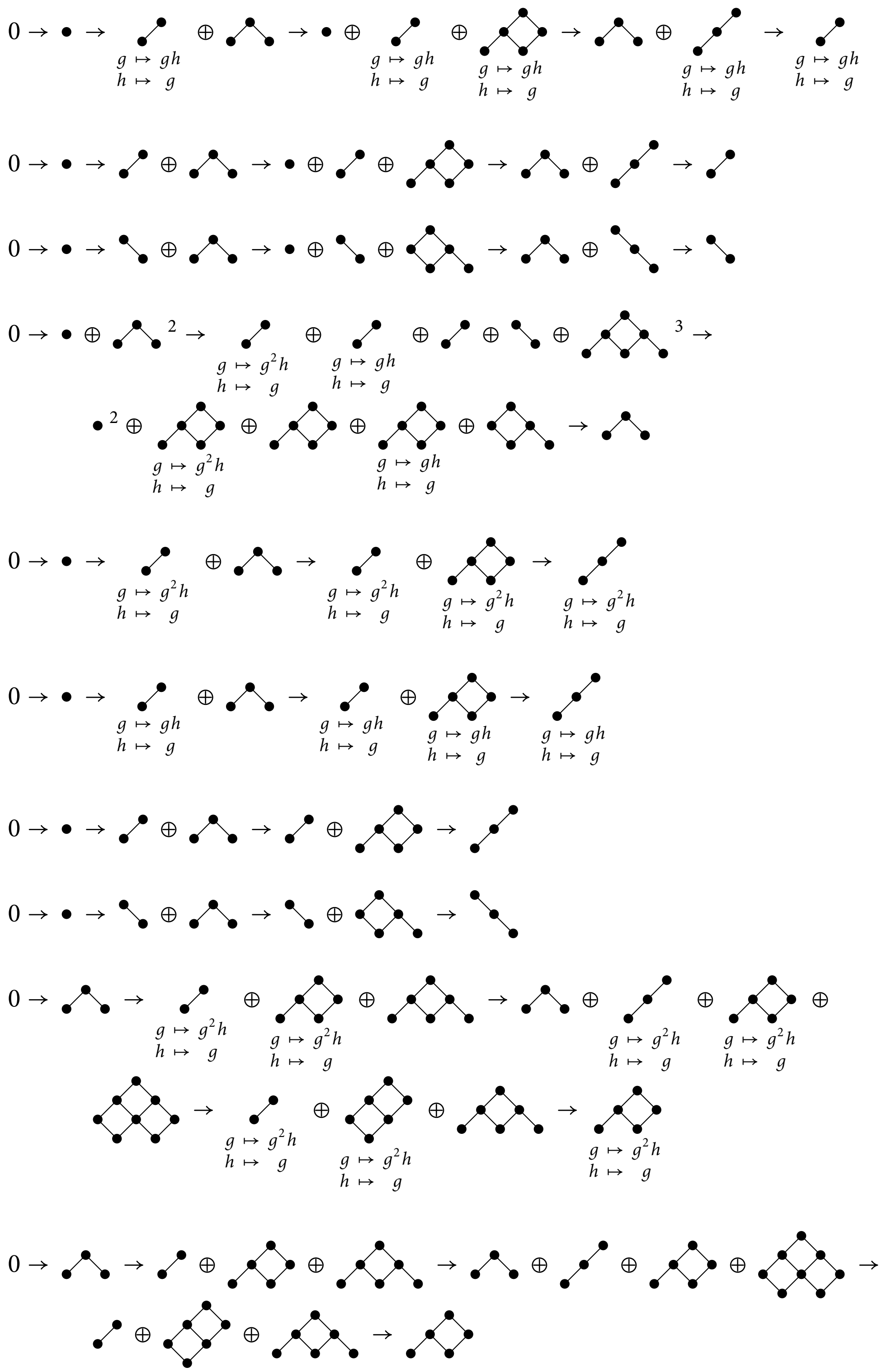
B.12 C_9

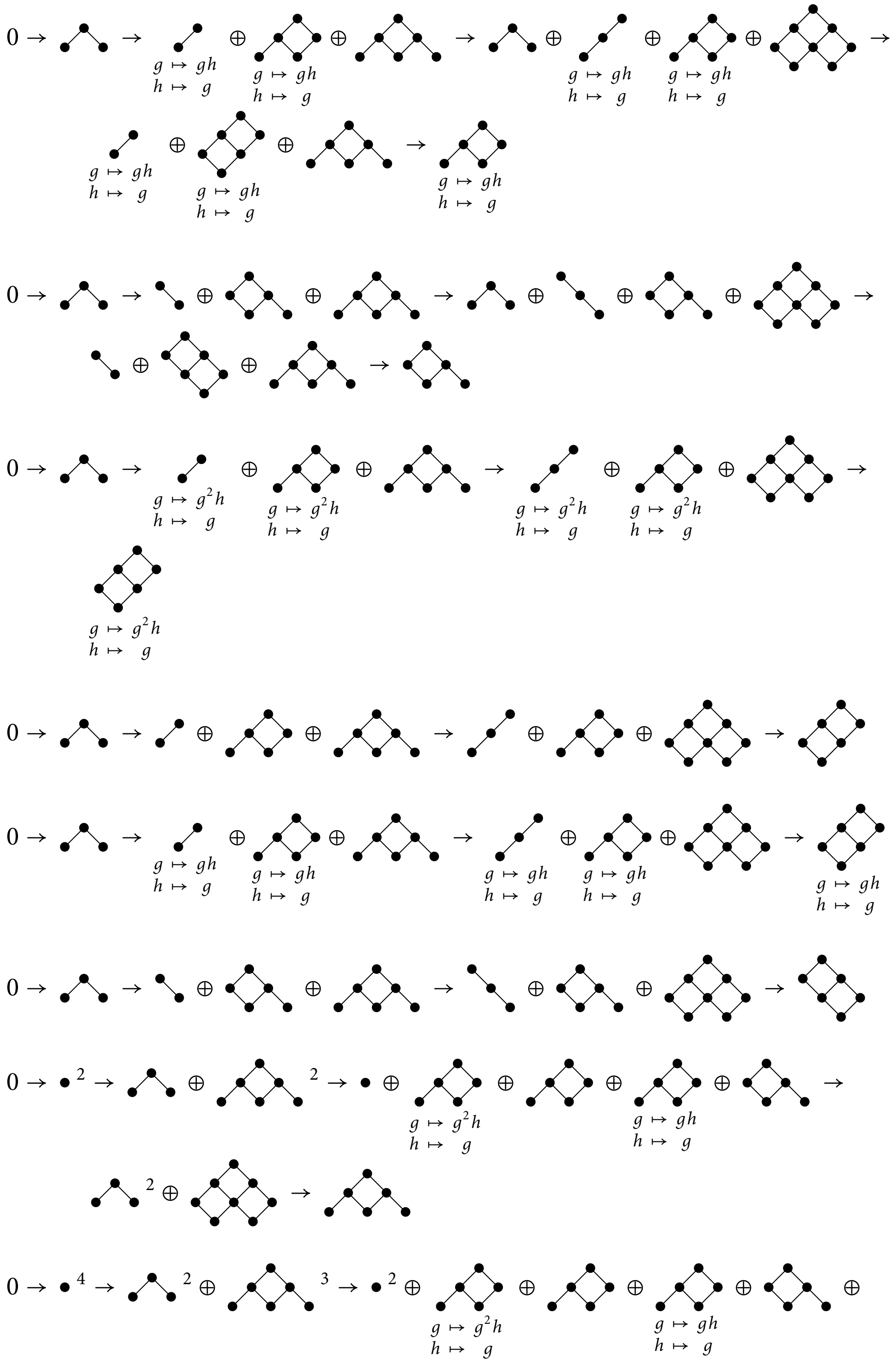


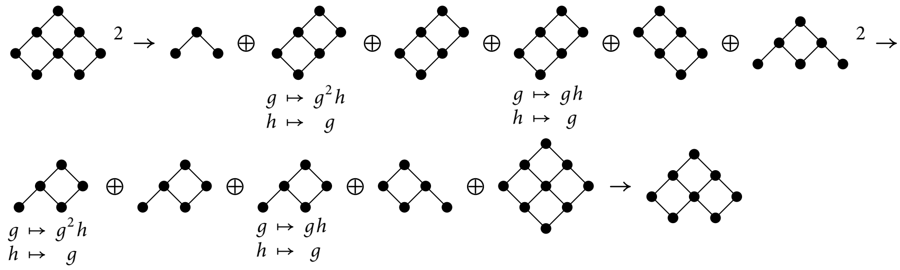


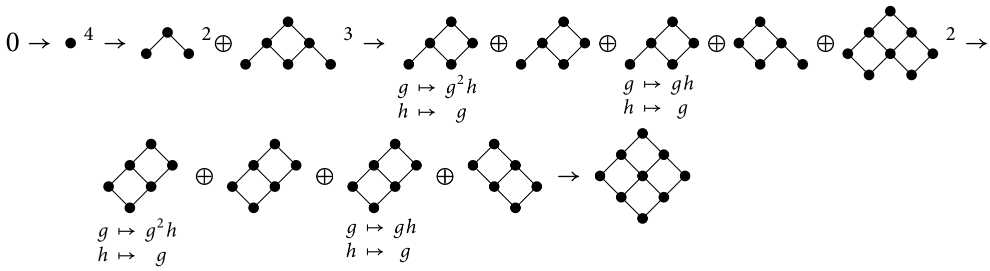
B.13 $C_3 \times C_3$




0 →
g ↦ gh
h ↦ g
g ↦ g^2h
h ↦ g
2
3


g ↦ gh
h ↦ g
g ↦ gh
h ↦ g
g ↦ gh
h ↦ g
g ↦ gh
h ↦ g
g ↦ gh
h ↦ g
g ↦ gh
h ↦ g
g ↦ gh
h ↦ g
g ↦ g²h
h ↦ g
g ↦ g²h
h ↦ g
g ↦ g²h
h ↦ g
g ↦ g²h
h ↦ g
g ↦ g²h
h ↦ g
g ↦ gh
h ↦ g
g ↦ gh
h ↦ g
g ↦ gh
h ↦ g
g ↦ gh
h ↦ g
g ↦ gh
h ↦ g
g ↦ g²h
h ↦ g
g ↦ gh
h ↦ g
g ↦ g²h
h ↦ g
g ↦ gh
h ↦ g


2 → ⊕ ⊕ ⊕ ⊕ ⊕ 2 →
g ↦ g²h
h ↦ g
g ↦ gh
h ↦ g
⊕ ⊕ ⊕ ⊕ →
g ↦ g²h
h ↦ g
g ↦ gh
h ↦ g


0 → 4 → 2⊕ 3 → ⊕ ⊕ ⊕ ⊕ 2 →
g ↦ g²h
h ↦ g
g ↦ gh
h ↦ g
⊕ ⊕ ⊕ →
g ↦ g²h
h ↦ g
g ↦ gh
h ↦ g

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