Representations of Quivers

MINGLE 2012

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Outline

Quiver

A *quiver*, Q, is a directed graph.

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Example

$$
\begin{tikzcd}
 & & & 4 \\
1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\zeta"'] & 2 \arrow[loop below, "\varepsilon"] \arrow[r, "\beta"] & 3 \arrow[ur, "\gamma"] \arrow[dr, "\delta"'] & \\
 & & & 5
\end{tikzcd}
$$

Representation

A *representation* of a quiver over a field k is an assignment of a k-vector space, V_i, to each vertex i and a k-linear map, $f_\alpha : V_i \to V_j$ to each edge $i \xrightarrow{\alpha} j$.

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$$\begin{tikzcd}
 & & & V_4 \\
V_1 \arrow[r, shift left, "f_\alpha"] \arrow[r, shift right, "f_\zeta"'] & V_2 \arrow[loop below, "f_\varepsilon"] \arrow[r, "f_\beta"] & V_3 \arrow[ur, "f_\gamma"] \arrow[dr, "f_\delta"'] & \\
 & & & V_5
\end{tikzcd}$$

Morphism of representations

If $V = (V_i, f_\alpha)$ and $W = (W_i, g_\alpha)$ are two representations of a quiver Q, then a *morphism* $\phi\colon V \to W$ is a set of k-linear maps $\{\phi_i\colon V_i \to W_i \mid i \text{ a vertex}\}$ such that for each edge $i \xrightarrow{\alpha} j$ the square

$$\begin{tikzcd} V_i \arrow[r, "f_\alpha"] \arrow[d, "\phi_i"'] & V_j \arrow[d, "\phi_j"] \\ W_i \arrow[r, "g_\alpha"'] & W_j \end{tikzcd}$$

commutes; that is $g_\alpha \phi_i = \phi_j f_\alpha$.

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If every ϕ_i is invertible then ϕ is an *isomorphism*.

Examples

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Basic linear algebra

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Jordan-Normal form

Algebra is x sighting

A ring[*], A, is an *algebra* over a field k, if A also has the structure of a k-vector space.

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$\mathbb{C}$ is a 2-dimensional algebra over $\mathbb{R}$.

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Paths

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Path algebra

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$\sum e_i$ is the identity element of kQ.

Module

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Example

$\mathbb{Z}^n$ is a module over $\mathbb{Z}$ with the obvious action:

$$(a_0, \ldots, a_n)x \mapsto (a_0 x, \ldots, a_n x)$$

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$$v_i \bullet e_j \quad \mapsto \quad \begin{cases} v_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad \qquad \text{for } v_i \in V_i$$

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$$v_i \bullet \alpha \mapsto \begin{cases} f_\alpha(v_i) & \text{if } i \xrightarrow{\alpha} \bullet \\ 0 & \text{otherwise} \end{cases}$$

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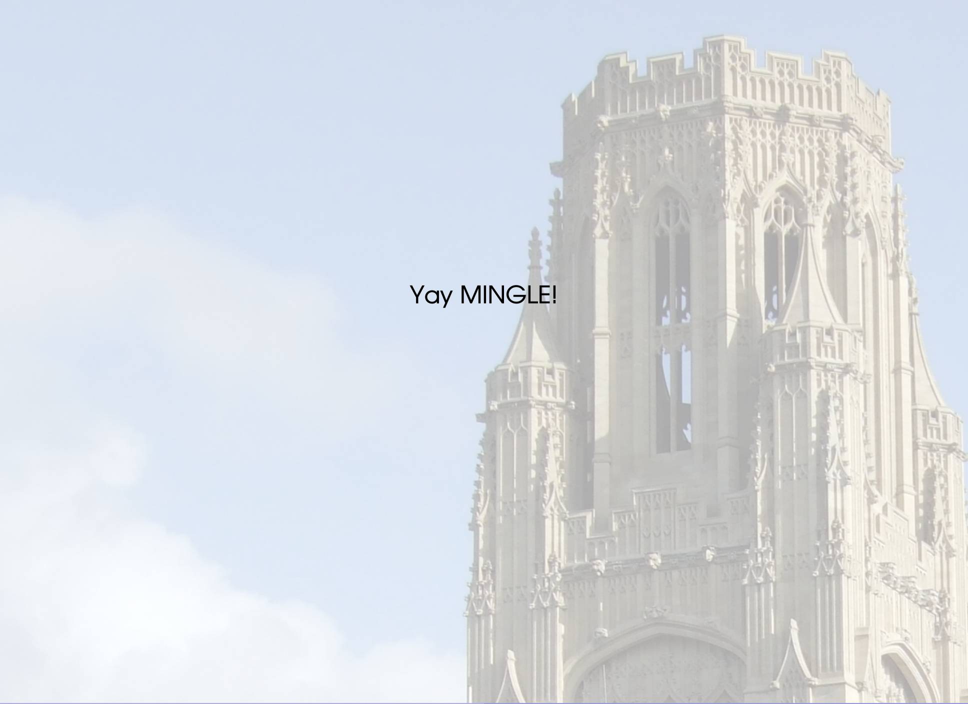

Yay MINGLE!