The Monster group and decomposition algebras

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Abstract

The classification of finite simple groups was a ground-breaking theorem in group theory, which states that every simple group (the prime numbers of group theory) belongs to one of three infinite families, or a list of 26 weird exceptions. The Monster group is the largest of these *sporadic groups*. When Griess first constructed the Monster he did so as a group acting on a real vector space, now known as the Griess algebra.

In this talk I will present some interesting properties of the Griess algebra, together with the definitions we draw generalizing these ideas (fusion laws and decomposition algebras). I will go on to show how the representation theory of groups can explain some of these properties. This naturally leads to some open questions into how group representation-theory and decomposition algebras are connected.

This work is joint with Tom De Medts, Sergey Shpectorov and Michiel Van Couwenberghe and the majority of our results are contained in [DMPSVC19]. I would also like to thank Justin McInroy for many useful conversations. I would also like to mention that many of these ideas originated with Sasha Ivanov and can be found in [Iva09].

[DMPSVC19] De Medts, Peacock, Shpectorov, and Van Couwenberghe, *Decomposition algebras and axial algebras*, arχiv preprint arXiv:1905.03481 (2019), 1–23

[Iva09] Ivanov, *The Monster group and Majorana involutions*, Cambridge Tracts in Mathematics, vol. 176, Cambridge University Press, Cambridge, 2009

1 Introduction

The classification of finite simple groups

The classification of finite simple groups was a ground-breaking result in group theory and was the culmination of decades of work. The simple groups can be thought of as the primes of group theory: in the same way the natural numbers can be broken down into their prime factors, so can a finite group be broken down into its simple constituents.

Theorem 1.1: *Classification of finite simple groups*

Every finite simple group is isomorphic to one of the following:

infinite families

- a cyclic group of prime order
- an alternating group of degree as least 5
- a group of Lie type

sporadic groups

*Or 26 and the Tits group if you are so inclined.

- one of 27* weird exceptions

The biggest sporadic group is called the Monster group, often denoted $\mathbb{M}$ and has an order of about 8×10^{53}. The second biggest is called the Baby Monster with an order of around 4×10^{33}.

History

Early to mid 70s

Bernd Fischer first predicted the existence of the Monster group in the early 70s and independently Robert Griess predicted its existence in a paper from 1976 ([Gri76]). Griess wrote that it should be a group with two conjugacy classes of involutions and the centralizer of one type of involution should be isomorphic to a double cover of the Baby Monster, $2.B$.

[Gri76] Griess, Jr., *The structure of the "Monster" simple group*, Proceedings of the Conference on Finite Groups (Univ. Utah, Park City, Utah, 1975), 1976, pp. 113–118

Late 70s

The character table of a group is a square grid of numerical data, which dictates much of the structure of the group, although not quite it's isomorphism class. Fischer, Donald Livingstone and Michael Thorne managed to calculate the character table of the Monster group ([Tho78]). Allegedly the calculations took all night and they were all heavy smokers, so much so that there were major complaints by the school's cleaning staff about the state of the room. Although, undeniably it was a remarkable achievement, not least because they still didn't know if the group even existed.

[Tho78] Thorne, *On Fischer's "Monster"*, Ph.D. thesis, University of Birmingham, 1978

The 80s

Griess finally managed to show that a Monster group existed in 1982 in his paper *The friendly giant* ([Gri82]). In this paper he exhibited the Monster group as the automorphism group of a 196 884-dimensional $\mathbb{R}$-algebra (now known as the Griess algebra), but at this time it was still not known to be unique. John Thompson had shown that uniqueness would follow from the existence of a 196 883-dimensional faithful representation and it was in 1985 that Norton first claimed to have proved of it's existence. Norton's proof never appeared in print however, and so it wasn't until 1989 when Griess, Ulrich Meierfrankenfeld and Yoav Segev published their paper [GMS89] that a proof of uniqueness was first seen.

[Gri82] Griess, Jr., *The friendly giant*, Invent. Math. **69** (1982), no. 1, 1–102

[GMS89] Griess, Jr., Meierfrankenfeld, and Segev, *A uniqueness proof for the Monster*, Ann. of Math. (2) **130** (1989), no. 3, 567–602

2 The Griess algebra

The Monster group was first constructed as the automorphism group of the 196, 884-dimensional $\mathbb{R}$-algebra now known as the Griess algebra.

Definition (Algebra)**.** For a field $\mathbb{F}$, an $\mathbb{F}$-algebra A is a vector space over $\mathbb{F}$ together with a multiplication that distributes over addition and commutes with scalars. That is there is a map

$$\begin{array}{ccc} A \times A & \longrightarrow & A \\ (a,b) & \mapsto & ab \end{array}$$

such that

$$\begin{gathered} a(b+c) = ab + ac \\ (a+b)c = ac + bc \\ (\lambda a)b = a(\lambda b) = \lambda(ab) \end{gathered}$$

for all $a, b, c \in A$ and $\lambda \in \mathbb{F}$.

Note that we can drop this last condition by instead defining the multiplication as a map from the tensor product, $A \otimes A \to A$, rather than from the Cartesian product.

The Griess algebra is a 196 884-dimensional $\mathbb{R}$-algebra, that is commutative but non-associative.

commutative $(ab) = (ba) \forall a, b \in A$,

non-associative $(ab)c \neq a(bc)$ for some $a, b, c \in A$.

Griess began with a particular module for the group $2^{1+24}.\mathrm{Co}_1$ and then defined the algebra multiplication on this module. He then showed that the group of automorphisms of this algebra formed a simple group consistent with the predicted Monster group.

Definition (Algebra automorphism)**.** For an algebra A, an automorphism $\varphi: A \to A$, is a linear map that respects the algebra multiplication:

$$\varphi(a)\varphi(b) = \varphi(ab) \qquad \text{for all } a, b \in A.$$

The Griess algebra turns out to be the same as the degree 2 part of the monster vertex operator algebra. In particular describes the conformal field theory for 24 free bosons compactified on the torus induced by the Leech lattice orbifolded by the two-element reflection group†.

†The author is far from ashamed to say that he has very little idea what this means, but thought it sounded important and so simply copied it directly from wikipedia.

Fusion laws

Inside the Griess algebra there are a set of special idempotent elements that we call axes. Letting A denote the Griess algebra and letting $a \in A$ be an axis we can define the adjoint action of a on A.

$$\begin{array}{rcl} \mathrm{ad}_a : A & \longmapsto & A \\ x & \mapsto & ax \end{array}$$

This defines a linear map on the vector space A and in particular it turns out to be diagonalizable. Thus we can decompose A into its eigenspaces. Since a is idempotent we know that $\mathrm{ad}_a(a) = a$ and thus a is a 1-eigenvector. In fact the 1-eigenspace is one-dimensional and hence, up to scaling, a is the only 1-eigenvector. The other eigenvalues for ad_a are 0, $1/4$ and $1/32$ and their eigenspaces have dimensions 96256, 4371 and 96256 respectively.

It may seem strange to have an idempotent with eigenvalues that are neither 1 or 0, but this is only implied in the associative case. If we have an associative multiplication and idempotent $e = e^2$ with x a λ-eigenvector for ad_e then

$$\lambda x = \mathrm{ad}_e(x) = ex = (e^2)x = e(ex) = e(\lambda x) = \lambda(ex) = \lambda^2 x$$

so $\lambda \in \{0, 1\}$. However in the non-associative case there is no reason to believe that $(e^2)x = e(ex)$ and hence λ is not so constrained.

We write A^a_λ for the λ-eigenspace of ad_a. Now we have already claimed that the 1-eigenspace is spanned by a and so it is clear that for any eigenvalue λ we have $A^a_1 A^a_\lambda \subseteq A^a_\lambda$. In fact the algebra multiplication also acts in a particularly nice way with respect to the other eigenspaces. This can be tabulated as follows

	A^a_1	A^a_0	$A^a_{1/4}$	$A^a_{1/32}$
A^a_1	A^a_1	0	$A^a_{1/4}$	$A^a_{1/32}$
A^a_0		A^a_0	$A^a_{1/4}$	$A^a_{1/32}$
$A^a_{1/4}$			$A^a_1 \oplus A^a_0$	$A^a_{1/32}$
$A^a_{1/32}$				$A^a_1 \oplus A^a_0 \oplus A^a_{1/4}$

For example if x and y are both $1/4$-eigenvectors then xy can be expressed as a sum of 1-eigenvectors and 0-eigenvectors. Note that since the multiplication is commutative we need only complete the upper triangular part of this table. This table inspires what we now call a *fusion law*.

Definition (Fusion law)**.** A *fusion law* is a set X and a binary operation $*$

$$\begin{array}{ccc} X \times X & \longrightarrow & 2^X \\ (x, y) & \mapsto & x * y \end{array}$$

If $x * y = y * x$ for all $x, y \in X$ then we say the law is *symmetric*.
If $e * x \subseteq \{x\}$ and $x * e \subseteq \{x\}$ for all $x \in X$ then we call e a *unit* of the fusion law.
If $e * x = \emptyset = x * e$ for all $x \in X$ then we say e is *annihilating*.
If $e * x \subseteq \{e\}$ and $x * e \subseteq \{e\}$ for all $x \in X$ then we say e is absorbing.

The fusion law we saw for the Griess algebra is called the *Ising fusion law*. It is a symmetric fusion law and both the 1 and 0 parts are units.

Definition (Morphism of fusion laws)**.** Let $(X, *)$ and $(Y, \star)$ be fusion laws. A morphism of $(X, *)$ to $(Y, \star)$ is a set map $\eta \colon X \to Y$ such that (for the obvious extension of η to the power sets)

$$\eta(x_1 * x_2) \subseteq \eta(x_1) \star \eta(x_2).$$

An obvious source of fusion laws come from the multiplication tables of groups. Specifically if G is a group we define the group fusion law $(G, \bullet)$ by $g \bullet h = \{gh\}$ for all $g, h \in G$. We call any morphism of a fusion law $(X, *)$ to the group fusion law for G a grading of $(X, *)$ by G.

Example. Let $X = \{1, 0, 1/4, 1/32\}$ and $(X, *)$ be the Ising fusion law. Define the map $\eta \colon X \to C_2$ from X to the cyclic group on two elements by

$$\eta(x) = \begin{cases} 0 & \text{if } x \in \{1, 0, 1/4\} \\ 1 & \text{if } x = 1/32 \end{cases}$$

	A^a_1	A^a_0	$A^a_{1/4}$	$A^a_{1/32}$
A^a_1	A^a_1	0	$A^a_{1/4}$	$A^a_{1/32}$
A^a_0		A^a_0	$A^a_{1/4}$	$A^a_{1/32}$
$A^a_{1/4}$			$A^a_1 \oplus A^a_0$	$A^a_{1/32}$
$A^a_{1/32}$				$A^a_1 \oplus A^a_0 \oplus A^a_{1/4}$

Decomposition algebras

From a fusion law we wish to define decompositions of an algebra with respect to the law.

Definition (Decomposition)**.** Given a fusion law $\mathcal{F} = (X, *)$ and an algebra A. An $\mathcal{F}$-decomposition of A is a direct sum decomposition of A as a vector space with a summand for each $x \in X$,

$$A = \bigoplus_{x \in X} A_x,$$

such that for all $x, y \in X$

$$A_x A_y \subseteq \bigoplus_{z \in x*y} A_z \stackrel{\text{def}}{=} A_{x*y}.$$

An $\mathcal{F}$-decomposition algebra is an algebra A together with a list of $\mathcal{F}$-decompositions Φ, indexed by some some set $\mathcal{I}$. That is, for each $i \in \mathcal{I}$, $\Phi[i]$ is an $\mathcal{F}$-decomposition.

Example. The Griess algebra is a decomposition algebra for the Ising fusion law, where we can take $\mathcal{I}$ to be the set of axes. For an axis $a \in A$, we define $\Phi[a]$ to be the eigenspace decomposition of A with respect to ad_a.

The final interesting property of the Griess algebra, over and above a decomposition algebra is the existence of an axis for each decomposition. This inspires the final definition of an axis for a decomposition.

Definition (Axis)**.** Let $A = \bigoplus_{x \in X} A_x$ be an $(X, *)$-decomposition of an algebra, A. An element $a \in A$ is called a *left axis* of the decomposition if there are field elements $\lambda_x \in \mathbb{F}$ for each $x \in X$ such that $ab = \lambda_x b$ whenever $b \in A_x$. We similarly define right axis for right multiplication by a and axis where the multiplication can be on either side, such as in the case of a commutative algebra.

A decomposition algebra A, is called (left/right) *axial* if there are scalars λ_x for each $x \in X$ and elements a_i for each $i \in \mathcal{I}$ such that a_i is a (left/right) axis for $\Phi[i]$.

3 Groups from algebras

The Griess algebra was developed as a way of describing the Monster group and in general we would like to assign a group to each decomposition algebra. Calculating the automorphism group of an algebra can be hard in practice and so we wish to find an easy to describe class of automorphisms. We begin with a decomposition algebra for a group fusion law. Note that everything we will describe can be transferred to a general fusion law via a grading.

Miyamoto automorphisms

For simplicity of exposition we will restrict to the case of a C_2-grading as is the case for the Griess algebra. Let A be a decomposition algebra for the C_2 group law and let $i \in \mathcal{I}$. We have the i^{th} decomposition of A into two parts $A = A_0^i \oplus A_1^i$. This decomposition allows us to define the i^{th} *Miyamoto automorphism* of A as

$$\begin{aligned} \tau_i \colon A &\longrightarrow A \\ x &\mapsto \begin{cases} x & x \in A_0^i \\ -x & x \in A_1^i \end{cases} \end{aligned}$$

and extended linearly.

It is easy to check that this is an automorphism: let $x_0, y_0 \in A_0$ and $x_1, y_1 \in A_1$ then for $x = x_0 + x_1$ and $y = y_0 + y_1$

$$\begin{aligned} \tau(xy) &= \tau(x_0y_0 + x_1y_0 + x_0y_1 + x_1y_1) \\ &= \tau(x_0y_0) + \tau(x_1y_0) + \tau(x_0y_1) + \tau(x_1y_1) && \text{by linearity} \\ &= x_0y_0 - x_1y_0 - x_0y_1 + x_1y_1 && \text{by the fusion law} \\ &= (x_0 - x_1)(y_0 - y_1) \\ &= \tau(x)\tau(y) \end{aligned}$$

The important point here is that the map

$$\begin{aligned} C_2 &\longrightarrow \mathbb{F} \\ 0 &\mapsto 1 \\ 1 &\mapsto -1 \end{aligned}$$

is a linear character for C_2. In general we can define a Miyamoto automorphism for any decomposition and any linear character.

Definition (Miyamoto group)**.** The *Miyamoto group* of a group-graded decomposition algebra is the subgroup of the automorphism group of the algebra, generated by all Miyamoto automorphisms.

4 Representation theory

We briefly saw reference to linear characters in the last section, this is an example of a 1-dimensional module (or representation) for a group. Here we will consider more general modules. We will restrict to the case of characteristic zero; in this setting the representation theory of groups is particularly nice: for a given finite group there is a finite number of simple modules, with all other modules being (isomorphic to) a direct sum of these. The character table for a group is a table numerical information linked to these simple modules. We begin with Schur's lemma, which essentially says that simple modules act like 1-dimensional spaces.

Definition (Module)**.** Let G be a group and $\mathbb{F}$ a field. A module M for G over $\mathbb{F}$ is an vector space (over $\mathbb{F}$) together with an action of G:

$$\begin{array}{ccc} G \times M & \longrightarrow & M \\ (g, m) & \mapsto & gm \end{array}$$

such that $g(hm) = (gh)m$ and $\lambda(gm) = g(\lambda m)$ for all $g, h \in G$, $m \in M$ and $\lambda \in \mathbb{F}$.

Lemma 4.1 (Schur)**.** *Let G be a finite group and let S and T be simple representations of G over a field $\mathbb{F}$. Then*

$$\mathrm{Hom}_{\mathbb{F}G}(S, T) \cong \begin{cases} \mathbb{F} & S \cong T \\ 0 & \textit{otherwise} \end{cases}$$

This means that once you have found an isomorphism pairing a basis of S to a basis of T, $s_i \mapsto t_i$, then the only other maps that can exist involve scaling the basis independently of i, $s_i \mapsto \lambda t_i$. A slight generalization of Schur's lemma tells us that if M is a module and S is a simple module then $\mathrm{Hom}(M, S) = 0$ unless S is isomorphic to a direct summand of M. If $\mathrm{Hom}(M, S) \neq 0$ then we say that S is a constituent of M.

The representation fusion law

The tensor product was mentioned briefly in the definition of a multiplication for an algebra. We can form the tensor product of modules in a similar way. If we take two modules M and N of dimensions m and n respectively, then their tensor product $M \otimes N$ is a module of dimension mn. This is simply the tensor product of vector spaces together with the diagonal action

$$g(x \otimes y) = gx \otimes gy.$$

Now since we can decompose $M \otimes N$ into simple summands we can ask what are its constituents. In particular, if S and T are simple modules we can ask what are the constituents of $S \otimes T$. This idea leads to the definition of the representation fusion law.

Definition (Representation fusion law)**.** Let $\mathcal{S} = \{S_i\}$ be a set of (some or all) simple modules for G over $\mathbb{F}$. The *representation fusion law* $(\mathcal{S}, *)$ is given by

$$S_i * S_j = \{T \in \mathcal{S} \mid T \text{ a constituent of } S_i \otimes S_j\}$$

Proposition 4.2. *Let G be a finite group, $\mathbb{F}$ a field and $\mathcal{S}$ the complete set of simple modules. Then the fusion law $(\mathcal{S}, *)$ is graded by the $Z(G)$, the center of G.*

The grading is given by first restricting $S \in \mathcal{S}$ to $Z(G) \leq G$, which has a unique constituent up to isomorphism. We then note that the character group of an abelian group is isomorphic to the group and this gives the grading.

Decomposition algebras via representation theory

We demonstrate how to get an (axial) decomposition algebra using representation theory.

Let A be the smallest faithful representation of the Monster over $\mathbb{C}$; this is a 196 883-dimensional module. We wish to turn the module A into an algebra and so the first thing we need is a multiplication

$$A \otimes A \to A.$$

As vector spaces there are many such maps however, as a result of Schur's lemma a group invariant map exists if an only if A is a constituent of $A \otimes A$. It turns out for this module there is a single copy of A in the direct sum decomposition of $A \otimes A$ and hence, up to scaling, there is a unique $\mathbb{M}$-invariant multiplication on A. This multiplication turns out to be commutative.

Next we wish to turn A into a decomposition algebra, and ideally we would like its fusion law to be graded. When Griess, first predicted the Monster he noted that $2.B$ should be contained inside it as the centralizer of a particular type of involution (the 2A involutions). Let $g \in \mathbb{M}$ be one of these involutions and $H = C_G(g) \cong 2.B$ be its centralizer. Importantly, H has a non-trivial center and so its representation fusion law is graded.

Let us consider the restriction of A to the subgroup H as a module. Although A is indecomposable as a $\mathbb{M}$ module $A\underset{H}{\downarrow}$ decomposes into four non-isomorphic simple summands

$$A\underset{H}{\downarrow} \cong A_a \oplus A_b \oplus A_c \oplus A_d$$

where A_a is the 1-dimensional trivial module and the remaining three modules are 96255-, 4371- and 96256-dimensional respectively. Let us consider the multiplication with respect to this decomposition. Firstly, since the multiplication is H-invariant (as it is $\mathbb{M}$-invariant) we know that $\mathrm{Hom}(A_i \otimes A_j, A_k) = 0$ unless A_k is a constituent of $A_i \otimes A_j$. Now since A_a is the trivial module we know immediately that $A_a \otimes A_i \cong A_i$ for any $i \in \{a, b, c, d\}$. Thus we can deduce that

$$A_aA_a \subseteq A_a \qquad A_aA_b \subseteq A_b \qquad A_aA_c \subseteq A_c \qquad A_aA_d \subseteq A_d$$

Similarly, we can consider the constituents of $A_i \otimes A_j$ for each pair $\{i, j\} \in \{b, c, d\}$ and deduce the following fusion table

	A_a	A_b	A_c	A_d
A_a	A_a	A_b	A_c	A_d
A_b		A_b	A_c	A_d
A_c			$A_a \oplus A_b$	A_d
A_d				$A_a \oplus A_b \oplus A_c$

This is exactly the fusion law for $\mathcal{S} = \{A_a, A_b, A_c, A_d\}$. Moreover, we know that $\mathrm{Hom}(A_a \otimes A_i, A_i) \cong \mathbb{C}$ for each $i \in \{a, b, c, d\}$ and hence if we let $e \in A_a$ then ad_e acts like a scalar on each part, that is e is an axis for the decomposition.

Finally, we want to consider the Miyamoto automorphism for this decomposition. In order to have a Miyamoto automorphism we require that the fusion law is graded. Proposition 4.2 tells us that the representation fusion law for H is C_2 graded, however we need to check that this subfusion law is also graded. For each simple module A_i we need to check which simple module is a constituent of $A_i\downarrow_{C_2}$. When we do this we see that A_a, A_b and A_c map to the trivial character and A_d maps to the non-trivial character. Thus we have a non-trivial Miyamoto automorphism that acts like the identity on the first three parts and acts like -1 on the fourth part. It is not difficult to show that this Miyamoto automorphism is the same as the action of g, the 2A involution we chose at the beginning.

Now the choice of g was arbitrary amongst its conjugacy class and by using all such conjugates we obtain a decomposition, axis and Miyamoto automorphism for each element in the 2A conjugacy class. Thus, for this particular axial algebra, the automorphism group and Miyamoto group are isomorphic, and are isomorphic to the Monster.

References

[DMPSVC19] Tom De Medts, Simon F. Peacock, Sergey Shpectorov, and Michiel Van Couwenberghe, *Decomposition algebras and axial algebras*, arχiv preprint arXiv:1905.03481 (2019), 1–23.

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