HW 1

(1) (a) Let $X = t^3$ and $Y = t^5$. Then we have

$$X^5 = t^{15} = Y^3,$$

and hence

$$\{(t^3, t^5) \mid t \in \mathbb{C}\} = \mathbb{V}(X^5 - Y^3) \subseteq \mathbb{C}^2.$$

(b) Firstly, note that since $t \mapsto t^2$ is a surjective map in $\mathbb{C}$, this set is equivalent to

$$\{(s, s^2) \mid s \in \mathbb{C}\}.$$

Then this set is clearly equal to

$$\mathbb{V}(Y - X^2) \subseteq \mathbb{C}^2.$$

(c) We have

$$\begin{aligned} &\{(t^2 - 2t, t^3 - 3t^2 + 3t + 2) \mid t \in \mathbb{C}\} \\ =&\{((t-1)^2 - 1, (t-1)^3 + 4) \mid t \in \mathbb{C}\} \\ =&\{(s^2 - 1, t^3 + 4) \mid s \in \mathbb{C}\}. \end{aligned}$$

Then if $X = s^2 - 1$ and $Y = t^3 + 4$, we have

$$(X+1)^3 = (Y-4)^2,$$

and hence the set is

$$\mathbb{V}((X+1)^3 - (Y-4)^2) \subseteq \mathbb{C}^2.$$

(d) For this question, set $X = t^3$ and $Y = t^4 + t^2$. We have

$$\begin{aligned} Y^2 &= t^8 + 2t^6 + t^4, \\ Y^3 &= t^{12} + 3t^{10} + 3t^8 + t^6, \\ X^2 Y &= t^{10} + t^8, \end{aligned}$$

and hence we have

$$Y^3 - 3X^2Y - X^4 - X^2 = 0.$$

So the set is

$$\mathbb{V}(Y^3 - 3X^2Y - X^4 - X^2) \subseteq \mathbb{C}^2.$$

(2) Parametrise the following varieties and make a guess at their dimension:

(a) The only $(x, y) \in \mathbb{C}^2$ with $x+y = x-y = 0$ is $(0,0)$. So the parametrisation is just $\{(0,0)\}$, and this has dimension 0.

(b) We have $X = Y^2$ and $Z = X^2 + Y$. So if we let $Y = t$, then from the first we have $X = t^2$, and then the second gives $Z = t^4 + t$. So the parametrisation is

$$\{(t^2, t, t^4 + t) \mid t \in \mathbb{C}\}.$$

This is one-dimensional.

(c) We have $Y = X^2 - Z^3$; beyond that, we have no more information. So we must use variables for both X and Z, giving a parametrisation
$$\{(s, s^2 - t^3, t) \mid s, t \in \mathbb{C}\}.$$
This is two-dimensional.

(d) First, note that the only solution with any component equal to 0 is the origin. The second two equations give that
$$X = 0 \;\Leftrightarrow\; Y = 0 \;\Leftrightarrow\; Z = 0;$$
now look at the first. If $W = 0$, this implies that one of X, Y, Z is zero (thus implying they all are), and if the X, Y, Z are zero, then $W = 0$. So assume W, X, Y, Z are all non-zero. Then we can write the first equation as
$$\frac{X}{W} = \frac{W}{YZ} =: t.$$
From this, we get
$$X = tW, \quad W = tYZ,$$
and hence satisfying the first part makes points of the form
$$(t^2YZ, Y, Z, tYZ).$$
Then using $X^2 = Y^3$, we get
$$t^4Y^2Z^2 = Y^3$$
and hence $Y = t^4Z^2$ (we can divide through by Y^2 since it is non-zero). So now we have points of the form
$$(t^6Z^3, t^4Z^2, Z, t^5Z^3).$$
Finally, using $Y^2 = Z^5$, we have that
$$t^8Z^4 = Z^5,$$
and hence $Z = t^8$. So we have a parametrisation
$$\{(t^{30}, t^{20}, t^8, t^{29}) \mid t \in \mathbb{C}\}.$$
This is one-dimensional.

(3) Describe the irreducible components of the following reducible varieties:

(a) We have
$$\mathbb{V}(XY) = \mathbb{V}(X) \cup \mathbb{V}(Y) \subseteq \mathbb{C}^2.$$
The lines $\mathbb{V}(X) \subseteq \mathbb{C}^2$ and $\mathbb{V}(Y) \subseteq \mathbb{C}^2$ are the irreducible components.

(b) We have
$$\mathbb{V}(XY) = \mathbb{V}(X) \cup \mathbb{V}(Y) \subseteq \mathbb{C}^3.$$
as before. However, $\mathbb{V}(X) \subseteq \mathbb{C}^3$ and $\mathbb{V}(Y) \subseteq \mathbb{C}^3$ are now *planes*, and these planes are the irreducible components.

(c) The first polynomial implies that either $X = 0$ or $Z = 0$. If $X = 0$, then the second polynomial implies $Y = 0$, and Z is free; so we have a component $\mathbb{V}(X, Y)$. This is an irreducible line.
If $Z = 0$, then the second polynomial is unchanged. This polynomial is zero if either $Y = 0$ or $X - 1 = 0$. So we have two more irreducible lines as components, namely $\mathbb{V}(Y, Z)$ and $\mathbb{V}(X - 1, Z)$.

(d) $\mathbb{V}(Y^2 - X^4, Y - ZX) \subseteq \mathbb{C}^3$

(4) We have

Ideal	Maximal?	Prime?	Radical?	Radical
(0)	X	X	X	(6)
(2)	✓	✓	✓	(2)
(6)	X	X	✓	(6)
(18)	X	X	X	(6)
(5)	X	X	✓	(5) = R

(5) Given an ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots ,$$

let

$$I = \bigcup_{n \in \mathbb{N}} I_n.$$

This I is an ideal of R, and so we have

$$I = (f_1, \ldots, f_m)$$

for some $f_i \in R$. Each f_i is in I_{n_i} for some $n_i \in \mathbb{N}$, and hence if $N = \max\{n_i\}$, then we have

$$I = I_N = I_{N+1} = \cdots .$$

(6) Suppose $I \neq (0)$, and let $a \in \mathbb{N}$ be its smallest positive element. Clearly we have $(a) \subseteq I$. Now suppose (for a contradiction) that there exists some $b \in \mathbb{Z}$ such that $b \in I$ but $b \notin (a)$.

We can then apply the Euclidean algorithm to find $q \in \mathbb{Z}$ and $0 < r < a$ such that

$$b = qa + r.$$

(We have $r > 0$ since $b \notin (a)$.) But then since $b \in I$ and $qa \in I$, we have $r = b - qa \in I$. So we've found an element $r \in \mathbb{N}$ of the ideal smaller than a, giving our contradiction.

(7) Recall that

$$\sqrt{(I)} \equiv \mathrm{Rad}(I) = \{r \mid r^n in I \text{ for some } n \in \mathbb{N}\}.$$

We need first to show that $\mathrm{Rad}(I)$ is an additive subgroup of R. Suppose $r, s \in \mathrm{Rad}(I)$. We want to show that $r+s \in \mathrm{Rad}(I)$ and that $-r \in \mathrm{Rad}(I)$. The second is easiest: suppose $r^m \in I$. Then we have

$$(-r)^{2m} = (-1)^{2m}(r^m)^2 = (r^m)^2 \in I,$$

and hence $-r \in \mathrm{Rad}(I)$. Now suppose that $s^n \in I$. We have

$$\begin{aligned}(r+s)^{m+n} &= \sum_{i=0}^{m+n} \binom{m+n}{i} r^i s^{m+n-i} \\ &= \sum_{i=0}^{m} \binom{m+n}{i} r^i s^{m+n-i} + \sum_{i=m+1}^{m+n} \binom{m+n}{i} r^i s^{m+n-i} \\ &= s^n \sum_{i=0}^{m} \binom{m+n}{i} r^i s^{m-i} + r^m \sum_{i=m+1}^{m+n} \binom{m+n}{i} r^{i-m} s^{m+n-i} \in I.\end{aligned}$$

So $r + s \in \mathrm{Rad}(I)$.

Now all we need to show that for $x \in R$ and $r \in \text{Rad}(I)$, we have $xr \in \text{Rad}(I)$. If $r^n \in I$, then we have

$$(xr)^n = x^n r^n,$$

and hence (since $x^n \in R$ and $r^n \in I$) we have $(xr)^n \in I$ and hence $xr \in \text{Rad}(I)$.

So $\text{Rad}(I)$ is an ideal as required.