HW 1

(1) Show that the following sets are varieties by writing them in the standard notation $\mathbb{V}(\{F_i\}_{i\in I}) \subset \mathbb{C}^n$.
  (a) $\{(t^3, t^5) \mid t \in \mathbb{C}\}$
  (b) $\{(t^2, t^4) \mid t \in \mathbb{C}\}$
  (c) $\{(t^2 - 2t, t^3 - 3t^2 + 3t + 2) \mid t \in \mathbb{C}\}$
  (d) $\{(t^3, t^4 + t^2) \mid t \in \mathbb{C}\}$

(2) Parametrise the following varieties and make a guess at their dimension:
  (a) $\mathbb{V}(X - Y, X + Y) \subseteq \mathbb{C}^2$
  (b) $\mathbb{V}(Y^2 - X, X^2 + Y - Z) \subseteq \mathbb{C}^3$
  (c) $\mathbb{V}(X^2 - Z^3 - Y) \subseteq \mathbb{C}^3$
  (d) $\mathbb{V}(XYZ - W^2, X^2 - Y^3, Y^2 - Z^5) \subseteq \mathbb{C}^4$

(3) Describe the irreducible components of the following reducible varieties:
  (a) $\mathbb{V}(XY) \subseteq \mathbb{C}^2$
  (b) $\mathbb{V}(XY) \subseteq \mathbb{C}^3$
  (c) $\mathbb{V}(XZ, Y - YX) \subseteq \mathbb{C}^3$
  (d) $\mathbb{V}(Y^2 - X^4, Y - ZX) \subseteq \mathbb{C}^3$

(4) Let $R = \mathbb{Z}/36\mathbb{Z}$. Determine whether the following ideals are prime, maximal or radical, and if they are not radical, determine their radical.
  (a) (0)
  (b) (2)
  (c) (6)
  (d) (18)
  (e) (5)

(5) Let R be a ring. Show that if every ideal of R is finitely generated, then R must be Noetherian.

(6) Show that $\mathbb{Z}$ is Noetherian. (Hint: take any ideal $I \subseteq \mathbb{Z}$, and show that it is generated by its smallest non-zero element.)

(7) Show that the radical $\sqrt{I}$ of an ideal I is an ideal.