Topics in modern geometry

Exercise sheet 2

Solution 1.
Let $a, b \in I_n$ so there are degree n polynomials $f, g \in I$ with leading coefficients a and b respectively. Then $f + g$ has leading coefficient $a + b$. If $r \in R$ then rf has leading coefficient fa. Finally, Xf is degree $n+1$ and has leading coefficient a.

Solution 2.
$\mathbb{V}(f) = \{(0,0)\}$. The polynomials that vanish at $(0,0)$ are exactly those with no constant term so $\mathbb{I}(\mathbb{V}(f)) = (X, Y)$.

Solution 3.
In $\mathbb{C}$, we have the factorisation $f = (X+i)(X-i)$ and so the two roots of f are $\pm i$. For a polynomial g:

$$g(i) = 0 \text{ if and only if } (X-i)|g$$
$$g(-i) = 0 \text{ if and only if } (X+i)|g$$

so

$$\text{so } g \in \mathbb{I}(\mathbb{V}(f)) \Leftrightarrow (X-i)|g \text{ and } (X+i)|g \Leftrightarrow f|g \Leftrightarrow g \in (f).$$

Solution 4.
Let $(x, y, z) \in \mathbb{V}(J)$. First note that $(x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + xz + yz)$ and so $x + y + z = 0$. Now if we substitute $z = -x - y$ into either equation we obtain $x^2 + xy - y^2 = 0$. Solving for y we have $y = \frac{1}{2}(-1 \pm i\sqrt{3})x$. Finally after calculating z we obtain the two lines parameterized by t

$$\left(1, \tfrac{1}{2}(-1 + i\sqrt{3}), \tfrac{1}{2}(-1 - i\sqrt{3})\right) t$$
$$\left(1, \tfrac{1}{2}(-1 - i\sqrt{3}), \tfrac{1}{2}(-1 + i\sqrt{3})\right) t$$

We have noted that if $(x, y, z) \in \mathbb{V}(J)$ then $x + y + z = 0$. Therefore $X + Y + Z \in \mathbb{I}(\mathbb{V}(J))$, but since $X + Y + Z$ is homogeneous of degree 1 and the generators of J are both homogeneous of degree 2, we can conclude that $X + Y + Z \notin J$.

Solution 5.

$W_1 \subseteq V = V_1 \cup \cdots \cup V_s$ so $W_1 = \bigcup_{i=1}^{s}(W_1 \cap V_i)$. As W_1 is irreducible we must have $W_1 = W_1 \cap V_i$ for some i, say (after reordering) $i = 1$, and then $W_1 \subseteq V_1$. Similarly for some j we must have $V_1 \subseteq W_j$, but as this implies $W_1 \subseteq W_j$ we must have $j = 1$ by the non-degeneracy. We have now shown that $V_1 = W_1$ and result follows by induction on s.

Solution 6.

Using the hint $K/\bar{k}$ is a field extension. Let $\lambda \in K$. If λ were transcendental over $\bar{k}$ then K would contain an isomorphic image of $k[X]$ contradicting $k \cong K$. Thus λ satisfies a minimal polynomial $f \in \bar{k}[X]$. Since $\bar{k}$ is algebraically closed $\lambda \in \bar{k}$ and hence φ is surjective.

Solution 7.

1. Let I be a prime ideal. If $f^m \in I$ for some integer m then either $f \in I$ or $f^{m-1} \in I$. Proceeding by induction we conclude that $f \in I$.

2. Let $M \subset R$ be a maximal ideal. Let $fg \in M$ but $f \notin M$. As $M \subsetneq (M, f)$, we can conclude that $1 \in (M, f)$. Therefore $g \in (Mg, fg) \subseteq M$.

Solution 8.

Let I be primary and $fg \in I$. Assume that $f \notin \operatorname{rad} I$ so that certainly $f \notin I$. Then $g^m \in I$ for some integer m and hence $g \in \operatorname{rad} I$.

Solution 9.

1. If $f \in I$ and $g \in J$ then $fg \in I$ and $fg \in J$. That is, $fg \in I \cap J$.

2. Let $x \in \mathbb{V}(IJ)$ so $f(x)g(x) = 0$ for all $f \in I$ and for all $g \in J$. If $f(x) = 0$ for all $f \in I$ then $x \in \mathbb{V}(I)$. If, on the other hand, there is some $f \in I$ for which $f(x) \neq 0$ then since $f(x)g(x) = 0$ for all $g \in J$ then $g(x) = 0$ for all $g \in J$, thus $x \in \mathbb{V}(J)$.

   Conversely, let $x \in \mathbb{V}(I) \cup \mathbb{V}(J)$. Either $f(x) = 0$ for all $f \in I$ or $g(x) = 0$ for all $g \in J$. Therefore $f(x)g(x) = 0$ for all $f \in I$ and for all $g \in J$. That is, $x \in \mathbb{V}(IJ)$.

3. If $x \in \mathbb{V}(I) \cup \mathbb{V}(J)$ then either $f(x) = 0$ for all $f \in I$ or $f(x) = 0$ for all $f \in J$ so certainly $f(x) = 0$ for all $f \in I \cap J$.

   Conversely, since $IJ \subseteq I \cap J$ by part 1 we have $\mathbb{V}(I \cap J) \subseteq \mathbb{V}(IJ) = \mathbb{V}(I) \cup \mathbb{V}(J)$ by part 2.

4. Take for example $I = J = (X) \subseteq \mathbb{R}[X]$. Then $IJ = (X^2) \neq (X) = I \cap J$.