Topics in modern geometry

Exercise sheet 3

Exercise 1.
Let $F : V \to W$ be an isomorphism of varieties.

1. Prove that if $V_1 \subseteq V$ is a subvariety then $W_1 = f(V_1) \subseteq W$ is a subvariety.
2. Show that $\dim_P V = \dim_{F(P)} W$ for any point $P \in V$.
3. Prove that if $\mathbb{C}[X_1, \dots, X_n]$ is isomorphic to $\mathbb{C}[Y_1, \dots, Y_m]$ then $m = n$.
   Hint: It may be helpful to use the correspondence between morphisms of varieties and homomorphisms of coordinate rings.

Exercise 2.
Consider the ideals

$$\begin{aligned} I_1 &= (XY + Y^2, XZ + YZ) \\ I_2 &= (XY + Y^2, XZ + YZ + XYZ + Y^2 Z) \\ I_3 &= (XY^2 + Y^3, XZ + YZ) \end{aligned}$$

1. Show that $I_1 = I_2 \supsetneq I_3$.
2. Show that $\mathbb{V}(I_1) = \mathbb{V}(I_3)$.
3. Which of these ideals are radical ideals?

Exercise 3.
Consider the map

$$\begin{aligned} \varphi \colon \mathbb{C} &\longrightarrow \mathbb{C}^4 \\ t &\mapsto (t^4, t^5, t^6, t^7) \end{aligned}$$

1. Show that $V = \varphi(\mathbb{C})$ is a variety.
2. Compute $T_0 V$.
3. Show that there is no variety $W \subseteq \mathbb{C}^3$ isomorphic to V. You may use that fact that an isomorphism of varieties induces an isomorphism of tangent spaces.

Exercise 4.
Find all the singular points of $\mathbb{V}(X^2 - YZ)$.