MATH20222: Introduction to Geometry

Sheet 1 — Semester 2 2020-21

Throughout $\mathbb{E}^n$ denotes an n-dimensional Euclidean vector space with orthonormal basis $\mathcal{B} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$.

1. Consider the following vectors in $\mathbb{R}^2$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix},$$

   (a) Show that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for $\mathbb{R}^2$.

   (b) Show that $\{\mathbf{a}, \mathbf{b}\}$ is a basis for $\mathbb{R}^2$.

   (c) Show that $\{\mathbf{e}_1, \mathbf{b}\}$ is *not* a basis for $\mathbb{R}^2$.

2. State whether each of the following maps $\langle -, - \rangle$ define an inner product on $\mathbb{R}^3$.

   [*where* $\mathbf{x} = (x_1, x_2, x_3)^\top, \mathbf{y} = (y_1, y_2, y_3)^\top$.]

   (a) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2$

   (b) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + 3x_2y_2 + 5x_3y_3$

   (c) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_2 + x_2y_1 + x_3y_3$

3. Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an (ordered) orthonormal basis of $\mathbb{E}^3$. Consider the ordered set of vectors $\mathcal{C} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ defined by $\mathcal{B}$ via:

   (a) $\mathbf{f}_1 = \mathbf{e}_2\,,\ \mathbf{f}_2 = \mathbf{e}_1\,,\ \mathbf{f}_3 = \mathbf{e}_3$

   (b) $\mathbf{f}_1 = \mathbf{e}_1\,,\ \mathbf{f}_2 = \mathbf{e}_1 + 3\mathbf{e}_3\,,\ \mathbf{f}_3 = \mathbf{e}_3$

   (c) $\mathbf{f}_1 = \mathbf{e}_1 - \mathbf{e}_2\,,\ \mathbf{f}_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2\,,\ \mathbf{f}_3 = \mathbf{e}_3$

   (d) $\mathbf{f}_1 = \mathbf{e}_2\,,\ \mathbf{f}_2 = \mathbf{e}_1\,,\ \mathbf{f}_3 = \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3$ where $\lambda \in \mathbb{R}$ is an arbitrary coefficient.

   For each set of vectors, write down the transition matrix from $\mathcal{B}$ to $\mathcal{C}$. Is $\mathcal{C}$ a basis? Is $\mathcal{C}$ orthogonal?

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4. Consider the sets of polynomials

$$V = \left\{ ax^2 + bx + c \;\middle|\; a,b,c \in \mathbb{R} \right\} \ , \ T = \left\{ x^2 + px + q \;\middle|\; p,q \in \mathbb{R} \right\}$$

with the natural operations of addition and multiplication of polynomials.

[*You may assume these operations satisfy commutativity, associativity and distributivity.*]

   (a) Which of these are vector spaces (over $\mathbb{R}$), and why?

   (b) Show the polynomials $1, x, x^2$ are linearly independent in V.

   (c) Calculate the dimension of V.

5. Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be vectors of a vector space V. Show that if at least one of the vectors is equal to $\mathbf{0}$, then they are linearly dependent.

6. Show that any three vectors in $\mathbb{R}^2$ must be linearly dependent.

7. Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be vectors of $\mathbb{E}^3$ such that each has unit length and they are pairwise orthogonal with each other. Show explicitly that they form a basis for $\mathbb{E}^3$.

8. Let $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ be vectors in $\mathbb{E}^3$ such that $\mathbf{a}, \mathbf{b}$ have unit length and are orthogonal to each other, and $\mathbf{c}$ has length $\sqrt{3}$ and forms the angle $\varphi = \arccos \frac{1}{\sqrt{3}}$ with $\mathbf{a}$ and $\mathbf{b}$.
Show that $\{\mathbf{a}, \mathbf{b}, \mathbf{c} - \mathbf{a} - \mathbf{b}\}$ forms an orthonormal basis for $\mathbb{E}^3$.

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