MATH20222: Introduction to Geometry

Sheet 2 — Semester 2 2020-21

1. Let $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y)$ be a basis for $\mathbb{R}^2$. Write down the matrix for the following linear operators in the basis $\mathcal{B}$.

(a)
$$\begin{aligned} P_a \colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto 2\mathbf{e}_x \\ \mathbf{e}_y &\mapsto 3\mathbf{e}_y \end{aligned}$$

(b)
$$\begin{aligned} P_b \colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto \mathbf{e}_x + \mathbf{e}_y \\ \mathbf{e}_y &\mapsto \mathbf{e}_x - \mathbf{e}_y \end{aligned}$$

(c)
$$\begin{aligned} P_c \colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto 2\mathbf{e}_x + \mathbf{e}_y \\ \mathbf{e}_y &\mapsto -4\mathbf{e}_x - 2\mathbf{e}_y \end{aligned}$$

(d)
$$\begin{aligned} P_d \colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto 2\mathbf{e}_y \\ \mathbf{e}_y &\mapsto \mathbf{e}_x + 2\mathbf{e}_y \end{aligned}$$

2. Let $P \colon V \to V$ be a linear operator acting on the vector space V and let $\mathcal{B} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be a basis for V. Prove that for any $\mathbf{v} \in V$, the column vector representing the image of $\mathbf{v}$ in the $\mathcal{B}$ basis, $[P(\mathbf{v})]_{\mathcal{B}}$, is given by the matrix multiplication: $[P]_{\mathcal{B}}\,[\mathbf{v}]_{\mathcal{B}}$.

   You may use without proof the fact that matrix multiplication is linear: that is, for a matrix M, column vectors v, w and scalars λ, μ the following holds:
$$M(\lambda v + \mu w) = \lambda M v + \mu M w$$

3. Let P be a linear operator acting on 2-dimensional vector space V and let $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y)$ be a basis for V, such that the matrix of P in the basis $\mathcal{B}$ is given by
$$[P]_{\mathcal{B}} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix}.$$
   Show that there is a basis $\mathcal{C} = (\mathbf{e}_u, \mathbf{e}_v)$ such that the linear operator in the basis $\mathcal{C}$ is
$$[P]_{\mathcal{C}} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}.$$
   Hint: *The second matrix means that $P(\mathbf{e}_u) = 4\mathbf{e}_u$ (it is an eigenvector). Use this fact to equate $[P(\mathbf{e}_u)]_{\mathcal{B}}$ with $[4\mathbf{e}_u]_{\mathcal{B}}$.*

4. Let $\mathcal{B}, \mathcal{C}$ be (ordered) bases of a vector space V. Show that the transition matrix ${}_{\mathcal{C}}T_{\mathcal{B}}$ from $\mathcal{C}$ to $\mathcal{B}$ is the inverse matrix of the transition matrix ${}_{\mathcal{B}}T_{\mathcal{C}}$ from $\mathcal{B}$ to $\mathcal{C}$ i.e.
$${}_{\mathcal{C}}T_{\mathcal{B}} = ({}_{\mathcal{B}}T_{\mathcal{C}})^{-1}.$$

5. Let $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y)$ be an orthonormal basis for 2-dimensional Euclidean space $\mathbb{E}^2$.

   (a) Consider the following alternative bases:

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(i) $\mathcal{C}_i = (\mathbf{e}_x, -\mathbf{e}_y)$ | (ii) $\mathcal{C}_{ii} = (\frac{\mathbf{e}_x+\mathbf{e}_y}{\sqrt{2}}, \frac{\mathbf{e}_y-\mathbf{e}_x}{\sqrt{2}})$

In each case write down the transition matrix from $\mathcal{B}$ to $\mathcal{C}_\bullet$ and calculate the transition matrix from $\mathcal{C}_\bullet$ to $\mathcal{B}$.

(b) For each of the linear operators P_a and P_b from question 1 calculate the matrix for the linear operator in each $\mathcal{C}$ basis above.

6. Let $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ be an orthonormal basis for $\mathbb{E}^3$. Calculate the determinant and trace for each of the following linear operators:

(a)
$$\begin{aligned} P_a\colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto \mathbf{e}_x + \mathbf{e}_y \\ \mathbf{e}_y &\mapsto \mathbf{e}_y + \mathbf{e}_z \\ \mathbf{e}_z &\mapsto \mathbf{e}_x + \mathbf{e}_z \end{aligned}$$

(b)
$$\begin{aligned} P_b\colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto \mathbf{e}_x - \mathbf{e}_y \\ \mathbf{e}_y &\mapsto \mathbf{e}_y - \mathbf{e}_z \\ \mathbf{e}_z &\mapsto \mathbf{e}_x - \mathbf{e}_z \end{aligned}$$

(c)
$$\begin{aligned} P_c\colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto \mathbf{e}_x \\ \mathbf{e}_y &\mapsto -\mathbf{e}_z \\ \mathbf{e}_z &\mapsto -\mathbf{e}_y \end{aligned}$$

(d)
$$\begin{aligned} P_d\colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto \sqrt{2}\mathbf{e}_x - \mathbf{e}_y + \mathbf{e}_z \\ \mathbf{e}_y &\mapsto \sqrt{2}(\mathbf{e}_y + \mathbf{e}_z) \\ \mathbf{e}_z &\mapsto \sqrt{2}\mathbf{e}_x + \mathbf{e}_y - \mathbf{e}_z \end{aligned}$$

7. For each linear operator in question 6:

(i) Determine if the operator is orthogonal or not.

Note: *It may be useful to recall a fact we learn about the determinant of an orthogonal operator.*

(ii) For $\lambda \in \mathbb{R}$ define the scaled linear operator $S\colon \mathbb{E}^3 \to \mathbb{E}^3$, by $S(\mathbf{v}) = \lambda P(\mathbf{v})$. Determine the values of λ (if any exist) such that S is an orthogonal operator.

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8. (a) Group the following bases into equivalent classes with respect to orientation. That is, all bases in the same class must share an orientation and any pair in different classes must have a different orientation.

$$\mathcal{B}_1 = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) \qquad \mathcal{B}_2 = (\mathbf{e}_y, \mathbf{e}_z, \mathbf{e}_x) \qquad \mathcal{B}_3 = (\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z)$$
$$\mathcal{B}_4 = (\mathbf{e}_x + \mathbf{e}_y, \mathbf{e}_y - \mathbf{e}_x, \mathbf{e}_z) \quad \mathcal{B}_5 = (\mathbf{e}_y + \mathbf{e}_z, \mathbf{e}_z - \mathbf{e}_y, \mathbf{e}_x) \quad \mathcal{B}_6 = (\mathbf{e}_x + \mathbf{e}_y, \mathbf{e}_x - \mathbf{e}_y, \mathbf{e}_z)$$

(b) Using your answer to part (a), or otherwise, determine which of the following linear operators preserve the orientation and which change the orientation.

$$P_1 \colon \mathbb{E}^3 \to \mathbb{E}^3 \quad P_1(\mathbf{e}_x) = \mathbf{e}_y, \quad P_1(\mathbf{e}_y) = \mathbf{e}_z, \quad P_3(\mathbf{e}_z) = \mathbf{e}_x$$
$$P_2 \colon \mathbb{E}^3 \to \mathbb{E}^3 \quad P_2(\mathbf{e}_x) = \mathbf{e}_y, \quad P_2(\mathbf{e}_y) = \mathbf{e}_x, \quad P_3(\mathbf{e}_z) = \mathbf{e}_z$$
$$P_3 \colon \mathbb{E}^3 \to \mathbb{E}^3 \quad P_3(\mathbf{e}_x) = \mathbf{e}_x + \mathbf{e}_y, \quad P_3(\mathbf{e}_y) = \mathbf{e}_y - \mathbf{e}_x, \quad P_3(\mathbf{e}_z) = \mathbf{e}_z$$
$$P_4 \colon \mathbb{E}^3 \to \mathbb{E}^3 \quad P_4(\mathbf{e}_x) = \mathbf{e}_y + \mathbf{e}_z, \quad P_4(\mathbf{e}_y) = \mathbf{e}_z - \mathbf{e}_y, \quad P_3(\mathbf{e}_z) = \mathbf{e}_x$$
$$P_5 \colon \mathbb{E}^3 \to \mathbb{E}^3 \quad P_5(\mathbf{e}_x) = \mathbf{e}_x + \mathbf{e}_y, \quad P_5(\mathbf{e}_y) = \mathbf{e}_x - \mathbf{e}_y, \quad P_3(\mathbf{e}_z) = \mathbf{e}_z$$

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