MATH20222: Introduction to Geometry

Sheet 3 — Semester 2 2020-21

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

Key exercises:

1. Consider the matrices

$$A = \begin{bmatrix} 5 & 1 \\ 4 & 5 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 3 \\ 6 & -6 & -6 \end{bmatrix}$$

   For each matrix:

   (a) find the eigenvalues λ by finding the roots of the polynomial $\det(M - \lambda I)$,

   (b) calculate the corresponding eigenvectors.

2. Let $\mathcal{B} = (\mathbf{e}, \mathbf{f}, \mathbf{g})$ be an orthonormal basis in $\mathbb{E}^3$. Define a linear operator P on $\mathbb{E}^3$ by

$$\begin{aligned} P(\mathbf{e}) &= \mathbf{e} \\ P(\mathbf{f}) &= \frac{\sqrt{2}}{2}\mathbf{f} + \frac{\sqrt{2}}{2}\mathbf{g} \\ P(\mathbf{g}) &= -\frac{\sqrt{2}}{2}\mathbf{f} + \frac{\sqrt{2}}{2}\mathbf{g} \end{aligned}$$

   (a) Write down the matrix $[P]_{\mathcal{B}}$.

   (b) Show P is an orthogonal operator that preserves orientation.

   (c) Find the axis and angle of rotation.

3. Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$ be an orthonormal basis for $\mathbb{E}^2$. Recall that the reflection operator R in the line spanned by $\mathbf{e}_1$ has matrix

$$[R]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

   Let $\mathbf{x}_\varphi = P_\varphi(\mathbf{e}_1)$ be the vector obtained by rotating $\mathbf{e}_1$ by φ. Let R_φ be the linear operator that is the reflection operator in the line spanned by $\mathbf{x}_\varphi$.

   (a) Write R_φ as a composition of rotation operators and the reflection operator R.

   (b) Show that $R_\varphi = Q_{2\varphi}$.

Department of Mathematics, University of Manchester

Simon Peacock: simon.peacock@manchester.ac.uk
Ben Smith: benjamin.smith-3@manchester.ac.uk

Extra exercises:

4. Let $\mathcal{B}$ be an orthonormal basis for $\mathbb{E}^2$, recall that P_φ is the operator that rotates by φ. Show the following identities for matrices of rotation operators in $\mathbb{E}^2$:

   (a) $\left[P_{-\varphi}\right]_{\mathcal{B}} = \left(\left[P_\varphi\right]_{\mathcal{B}}\right)^{\mathsf{T}} = \left(\left[P_\varphi\right]_{\mathcal{B}}\right)^{-1}$

   (b) $\left[P_\varphi\right]_{\mathcal{B}}[P_\theta]_{\mathcal{B}} = \left[P_{\varphi+\theta}\right]_{\mathcal{B}}$

5. Let $\mathcal{B} = (\mathbf{e}, \mathbf{f}, \mathbf{g})$ be an orthonormal basis in $\mathbb{E}^3$. Consider the linear operator P_1 in $\mathbb{E}^3$ defined by

   $$P_1(\mathbf{e}) = \mathbf{f} \qquad P_1(\mathbf{f}) = \mathbf{e} \qquad P_1(\mathbf{g}) = \mathbf{g}.$$

   Also consider the linear operator P_2 that is the reflection operator in the plane spanned by $\mathbf{e}$ and $\mathbf{f}$.

   (a) State whether P_1 and P_2 preserves orientation.

   (b) Define the operator $P = P_1 \circ P_2$ that applies the operator P_2 followed by the operator P_1. Does P preserve orientation?

   (c) Show that P is a rotation operator. Find its axis and angle of rotation.

6. Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$ be an orthonormal basis for $\mathbb{E}^2$. Consider the linear operators P_1, P_2 on $\mathbb{E}^2$ defined by

   $$P_1(\mathbf{e}_1) = \mathbf{e}_1 \qquad P_2(\mathbf{e}_1) = \mathbf{e}_1 - \mathbf{e}_2$$
   $$P_1(\mathbf{e}_2) = \mathbf{e}_1 + \mathbf{e}_2 \qquad P_2(\mathbf{e}_2) = \mathbf{e}_2$$

   (a) Write down the matrices for P_1, P_2. Which of them are orthogonal operators?

   (b) Find all linear operators of the form $P = aP_1 + bP_2$ that are orthogonal (where $a, b \in \mathbb{R}$).

   (c) For each orthogonal P, write it as either a rotation operator P_φ or a reflection operator Q_φ.

Simon Peacock: simon.peacock@manchester.ac.uk
Ben Smith: benjamin.smith-3@manchester.ac.uk

7. Let $\mathbf{n} \in \mathbb{E}^3$ be a unit vector and consider the following operators on $\mathbb{E}^3$

$$P_1(\mathbf{x}) = \mathbf{x} - 2\langle \mathbf{n}, \mathbf{x} \rangle \mathbf{n} \quad , \quad P_2(x) = 2\langle \mathbf{n}, \mathbf{x} \rangle \mathbf{n} - \mathbf{x}.$$

   (a) Show both operators are orthogonal.

   (b) Show the first operator is a reflection, along with the plane it is a reflection through.

   (c) Show the second operator is a rotation. Find its axis and angle of rotation.

   [(Almost) all of this question can be done without writing a matrix! You may find it helpful to consider the plane $H_{\mathbf{n}}$ whose normal vector is $\mathbf{n}$:[1]]

$$H_{\mathbf{n}} = \left\{ \mathbf{x} \in \mathbb{E}^3 \,\middle|\, \langle \mathbf{n}, \mathbf{x} \rangle = 0 \right\}$$

[1] In general, any plane in $\mathbb{E}^3$ (or $(n-1)$-dimensional *hyperplane* in $\mathbb{E}^n$) can be written as the set of vectors orthogonal to a normal vector $\mathbf{n}$.

Department of Mathematics, University of Manchester
Simon Peacock: simon.peacock@manchester.ac.uk
Ben Smith: benjamin.smith-3@manchester.ac.uk