MATH20222: Introduction to Geometry

Sheet 4 — Semester 2 2020-21

Key exercises:

1. Calculate the area of the parallelogram $\varnothing(\mathbf{a}, \mathbf{b})$, formed by the following vectors, where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an orthonormal basis.

   (a) $\mathbf{a} = 2\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$ and $\mathbf{b} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$;

   (b) $\mathbf{a} = 5\mathbf{e}_1 + 8\mathbf{e}_2 + 4\mathbf{e}_3$ and $\mathbf{b} = 10\mathbf{e}_1 + 16\mathbf{e}_2 + 8\mathbf{e}_3$.

2. Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$ be an orthonormal basis for $\mathbb{E}^2$ and let P be a linear operator with the matrix: $[P]_{\mathcal{B}} = \left[\begin{smallmatrix} 5 & -1 \\ 2 & 2 \end{smallmatrix}\right]$. (See Exercise Sheet 2 Q3.)

   (a) Calculate the area of the parallelogram $\varnothing(\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + 2\mathbf{e}_2)$.

   (b) Recall the $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_2$ is an eigenvector of P with eigenvalue 4; and $\mathbf{b} = \mathbf{e}_1 + 2\mathbf{e}_2$ is an eigenvector of P with eigenvalue 3. Calculate the area of the image parallelogram $\varnothing(P(\mathbf{a}), P(\mathbf{b}))$.

   (c) Compare your answers to parts (a) and (b) with the $\det P$.

3. Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an orthonormal basis for $\mathbb{E}^3$. Let P be the linear operator with matrix $[P]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 & -4 \\ 2 & 4 & -4 \\ 0 & 0 & 1 \end{bmatrix}$.

   (a) Calculate the volume of the parallelepiped formed by the vectors $\mathbf{a} = 2\mathbf{e}_1 + \mathbf{e}_3$; $\mathbf{b} = 4\mathbf{e}_2 + 3\mathbf{e}_3$; and $\mathbf{c} = \mathbf{e}_1 - \mathbf{e}_2$.

   (b) Calculate $P(\mathbf{a})$, $P(\mathbf{b})$ and $P(\mathbf{c})$.

   (c) What is the volume of the parallelepiped $\varnothing(P(\mathbf{a}), P(\mathbf{b}), P(\mathbf{c}))$?

   (d) Compare the volumes from parts (a) and (c) with the determinant of P.

Department of Mathematics, University of Manchester
Simon Peacock: simon.peacock@manchester.ac.uk
Ben Smith: benjamin.smith-3@manchester.ac.uk

Extra exercises:

4. Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an orthonormal basis in $\mathbb{E}^3$. Find a unit vector $\mathbf{n}$, such that the following conditions hold:

   1. $\mathbf{n}$ is orthogonal to $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$;
   2. $\mathbf{n}$ is orthogonal to $\mathbf{b} = \mathbf{e}_1 + 3\mathbf{e}_2 + 2\mathbf{e}_3$;
   3. The basis $(\mathbf{a}, \mathbf{b}, \mathbf{n})$ has an orientation opposite to $\mathcal{B}$.

   Express $\mathbf{n}$ using the basis $\mathcal{B}$.

5. Elisa and Faraz calculate the vector product of two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{E}^3$. Elisa uses the orthonormal basis $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and Faraz use the orthonormal basis $\mathcal{F} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$. The vectors expressed in these bases are:

$$[\mathbf{a}]_{\mathcal{E}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad [\mathbf{b}]_{\mathcal{E}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad [\mathbf{a}]_{\mathcal{F}} = \begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix} \qquad [\mathbf{b}]_{\mathcal{F}} = \begin{bmatrix} b'_1 \\ b'_2 \\ b'_3 \end{bmatrix}$$

   They both use the *determinant formula* for calculations:

$$\mathbf{a} \times \mathbf{b} \stackrel{?}{=} \underbrace{\det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}}_{\text{Elisa's calculations}} \stackrel{?}{=} \underbrace{\det \begin{bmatrix} \mathbf{f}_1 & \mathbf{f}_2 & \mathbf{f}_3 \\ a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \end{bmatrix}}_{\text{Faraz's calculations}}.$$

   (a) In what situations do their calculations give the same vector, and when do they give different answers?

   (b) In each case what can you say about the linear operator P, that maps each vector $\mathbf{e}_i \mapsto \mathbf{f}_i$?

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6. Let $\mathcal{B}$, P, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be as in question 3.

   (a) What is the area of the parallelograms:

   (i) $\square(\mathbf{a}, \mathbf{b})$; (ii) $\square(\mathbf{a}, \mathbf{c})$?

   (b) Compare these to the areas of the image parallelograms:

   (i) $\square(P(\mathbf{a}), P(\mathbf{b}))$; (ii) $\square(P(\mathbf{a}), P(\mathbf{c}))$?

   (c) Can you say anything about the determinant of a linear operator acting on a 3-dimensional space and the scaling of parallelograms?

   (d) Can you find a linear operator that fixes the volume of parallelepipeds but scales the area of some parallelogram by λ?

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7.[†] In this question we will prove the determinant formula satisfies the axioms of a vector product. Fix an orthogonal basis $\mathcal{B}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ for $\mathbb{E}^3$, which we use to define the orientation of $\mathbb{E}^3$. For any two vectors $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$ and $\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3$ define the function $- \times - \colon \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{E}^3$ by the determinant formula:

$$\mathbf{v} \times \mathbf{w} = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$

(VP-AC) Show, or explain why, $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$ for all vectors $\mathbf{v}$ and $\mathbf{w}$.

(VP-⊥) (i) Show that $\langle \mathbf{v} \times \mathbf{w}, \mathbf{v} \rangle = 0$ for all vectors $\mathbf{v}$ and $\mathbf{w}$.

(ii) Using (VP-AC) deduce that $\langle \mathbf{v} \times \mathbf{w}, \mathbf{w} \rangle = 0$.

(VP-Lin) (i) Show the following identity holds:

$$\det \begin{bmatrix} \lambda a + \mu a' & \lambda b + \mu b' \\ c & d \end{bmatrix} = \lambda \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \mu \det \begin{bmatrix} a' & b' \\ c & d \end{bmatrix}.$$

(ii) Prove that $(\lambda \mathbf{v} + \mu \mathbf{w}) \times \mathbf{x} = \lambda(\mathbf{v} \times \mathbf{x}) + \mu(\mathbf{w} \times \mathbf{x})$ for all vectors $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{x}$.

(VP-Len) Recall the identity of Lemma 1.83: $|\mathbf{v} \times \mathbf{w}|^2 + \langle \mathbf{v}, \mathbf{w} \rangle^2 = |\mathbf{v}|^2 |\mathbf{w}|^2$.

Using this formula, or otherwise, prove $|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}||\mathbf{w}|$ for all perpendicular vectors $\mathbf{v}$ and $\mathbf{w}$.

(VP-O) Let $v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$ and $\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3$, be linearly independent vectors.

(i) Let $\mathcal{C} = (\mathbf{v}, \mathbf{w}, \mathbf{v} \times \mathbf{w})$, which you may assume is a basis for $\mathbb{E}^3$. Write down the transition matrix T, from basis $\mathcal{B}$ to basis $\mathcal{C}$.

(ii) Calculate a formula for the determinant of T and deduce that $\mathcal{B}$ and $\mathcal{C}$ have the same orientation.

It may be helpful to use the 3[rd] *column to calculate the determinant:*

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix} - f \det \begin{bmatrix} a & b \\ g & h \end{bmatrix} + i \det \begin{bmatrix} a & b \\ d & e \end{bmatrix}.$$

Finally: Deduce that $- \times -$ is a vector product as defined in Definition 1.77.

Simon Peacock: simon.peacock@manchester.ac.uk
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