MATH20222: Introduction to Geometry

Sheet 2 Solutions — Semester 2 2020-21

1. Let $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y)$ be a basis for $\mathbb{R}^2$. Write down the matrix for the following linear operators in the basis $\mathcal{B}$.

(a)
$$\begin{aligned} P_a \colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto 2\mathbf{e}_x \\ \mathbf{e}_y &\mapsto 3\mathbf{e}_y \end{aligned}$$

(b)
$$\begin{aligned} P_b \colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto \mathbf{e}_x + \mathbf{e}_y \\ \mathbf{e}_y &\mapsto \mathbf{e}_x - \mathbf{e}_y \end{aligned}$$

(c)
$$\begin{aligned} P_c \colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto 2\mathbf{e}_x + \mathbf{e}_y \\ \mathbf{e}_y &\mapsto -4\mathbf{e}_x - 2\mathbf{e}_y \end{aligned}$$

(d)
$$\begin{aligned} P_d \colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto 2\mathbf{e}_y \\ \mathbf{e}_y &\mapsto \mathbf{e}_x + 2\mathbf{e}_y \end{aligned}$$

Solution:

(a) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}$

2. Let $P \colon V \to V$ be a linear operator acting on the vector space V and let $\mathcal{B} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be a basis for V. Prove that for any $\mathbf{v} \in V$, the column vector representing the image of $\mathbf{v}$ in the $\mathcal{B}$ basis, $[P(\mathbf{v})]_{\mathcal{B}}$, is given by the matrix multiplication: $[P]_{\mathcal{B}} [\mathbf{v}]_{\mathcal{B}}$.

You may use without proof the fact that matrix multiplication is linear: that is, for a matrix M, column vectors v, w and scalars λ, μ the following holds:

$$M(\lambda v + \mu w) = \lambda M v + \mu M w$$

Solution: Fix an arbitrary vector $\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i$. We have

$$P(\mathbf{v}) = P\left(\sum_{i=1}^{n} v_i \mathbf{e}_i\right) = \sum_{i=1}^{n} v_i P(\mathbf{e}_i) \qquad \text{(linearity of } P\text{)}$$

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therefore by linearity of the notation $[-]_{\mathcal{B}}$ we also have

$$\begin{aligned}
[P(\mathbf{v})]_{\mathcal{B}} &= \sum_{i=1}^{n} v_i [P(\mathbf{e}_i)]_{\mathcal{B}} \\
&= \sum_{i=1}^{n} v_i [P]_{\mathcal{B}} [\mathbf{e}_i]_{\mathcal{B}} && ([P(\mathbf{e}_i)]_{\mathcal{B}} \text{ is the } i^{\text{th}} \text{ column of } [P]_{\mathcal{B}}) \\
&= [P]_{\mathcal{B}} \left(\sum_{i=1}^{n} v_i [\mathbf{e}_i]_{\mathcal{B}} \right) && (\text{linearity of matrix multiplication}) \\
&= [P]_{\mathcal{B}} \left[\left(\sum_{i=1}^{n} v_i \mathbf{e}_i \right) \right]_{\mathcal{B}} && (\text{linearity of } [-]_{\mathcal{B}}) \\
&= [P]_{\mathcal{B}} [\mathbf{v}]_{\mathcal{B}}
\end{aligned}$$

3. Let P be a linear operator acting on 2-dimensional vector space V and let $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y)$ be a basis for V, such that the matrix of P in the basis $\mathcal{B}$ is given by

$$[P]_{\mathcal{B}} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix}.$$

Show that there is a basis $\mathcal{C} = (\mathbf{e}_u, \mathbf{e}_v)$ such that the linear operator in the basis $\mathcal{C}$ is

$$[P]_{\mathcal{C}} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}.$$

Hint: *The second matrix means that* $P(\mathbf{e}_u) = 4\mathbf{e}_u$ *(it is an eigenvector). Use this fact to equate* $[P(\mathbf{e}_u)]_{\mathcal{B}}$ *with* $[4\mathbf{e}_u]_{\mathcal{B}}$.

Solution: The second matrix tells us that $P(\mathbf{e}_u) = 4\mathbf{e}_u$. We can write $\mathbf{e}_u$ in the $\mathcal{B}$ basis: $\mathbf{e}_u = \lambda \mathbf{e}_x + \mu \mathbf{e}_y$. Then

$$\begin{bmatrix} 4\lambda \\ 4\mu \end{bmatrix} = [4\mathbf{e}_u]_{\mathcal{B}} = [P(\mathbf{e}_u)]_{\mathcal{B}} = [P]_{\mathcal{B}} [\mathbf{e}_u]_{\mathcal{B}} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} 5\lambda - \mu \\ 2\lambda + 2\mu \end{bmatrix}$$

so $\lambda = \mu$.

Similarly we see that $P(\mathbf{e}_v) = 3\mathbf{e}_v$, and if $\mathbf{e}_v = \alpha \mathbf{e}_x + \beta \mathbf{e}_y$ we have

$$\begin{bmatrix} 3\alpha \\ 3\beta \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 5\alpha - \beta \\ 2\alpha + 2\beta \end{bmatrix}$$

so $2\alpha = \beta$. We have two degrees of freedom to choose the basis, one such choice is $\mathcal{C} = (\mathbf{e}_x + \mathbf{e}_y, \mathbf{e}_x + 2\mathbf{e}_y)$.

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The transition matrix ${}_{\mathcal{B}}T_{\mathcal{C}} = \left[\begin{smallmatrix}1 & 1\\ 1 & 2\end{smallmatrix}\right]$, with inverse $\left[\begin{smallmatrix}2 & -1\\ -1 & 1\end{smallmatrix}\right]$ and we may check that

$$\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}^{-1}\begin{bmatrix}5 & -1\\ 2 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix} = \begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}4 & 3\\ 4 & 6\end{bmatrix} = \begin{bmatrix}4 & 0\\ 0 & 3\end{bmatrix}$$

4. Let $\mathcal{B}, \mathcal{C}$ be (ordered) bases of a vector space V. Show that the transition matrix ${}_{\mathcal{C}}T_{\mathcal{B}}$ from $\mathcal{C}$ to $\mathcal{B}$ is the inverse matrix of the transition matrix ${}_{\mathcal{B}}T_{\mathcal{C}}$ from $\mathcal{B}$ to $\mathcal{C}$ i.e.

$$ {}_{\mathcal{C}}T_{\mathcal{B}} = ({}_{\mathcal{B}}T_{\mathcal{C}})^{-1}.$$

Solution: We can solve this very quickly by using the transitivity property of transition matrices from Lemma 1.37 in the notes. In particular, we have that ${}_{\mathcal{C}}T_{\mathcal{B}}\,{}_{\mathcal{B}}T_{\mathcal{C}} = {}_{\mathcal{C}}T_{\mathcal{C}}$. The transition matrix from $\mathcal{C}$ to $\mathcal{C}$ is just the identity matrix, therefore

$$ {}_{\mathcal{C}}T_{\mathcal{B}} = {}_{\mathcal{C}}T_{\mathcal{B}}I_n = {}_{\mathcal{C}}T_{\mathcal{B}}\,{}_{\mathcal{B}}T_{\mathcal{C}}\,({}_{\mathcal{B}}T_{\mathcal{C}})^{-1} = I_n({}_{\mathcal{B}}T_{\mathcal{C}})^{-1} = ({}_{\mathcal{B}}T_{\mathcal{C}})^{-1}.$$

5. Let $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y)$ be an orthonormal basis for 2-dimensional Euclidean space $\mathbb{E}^2$.

   (a) Consider the following alternative bases:

   (i) $\mathcal{C}_i = (\mathbf{e}_x, -\mathbf{e}_y)$ | (ii) $\mathcal{C}_{ii} = (\frac{\mathbf{e}_x+\mathbf{e}_y}{\sqrt{2}}, \frac{\mathbf{e}_y-\mathbf{e}_x}{\sqrt{2}})$

   In each case write down the transition matrix from $\mathcal{B}$ to $\mathcal{C}_\bullet$ and calculate the transition matrix from $\mathcal{C}_\bullet$ to $\mathcal{B}$.

   (b) For each of the linear operators P_a and P_b from question 1 calculate the matrix for the linear operator in each $\mathcal{C}$ basis above.

Solution:

(a) Using the fact that ${}_{\mathcal{C}}T_{\mathcal{B}} = {}_{\mathcal{B}}T_{\mathcal{C}}{}^{-1}$

(i) $${}_{\mathcal{B}}T_{\mathcal{C}_i} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \qquad {}_{\mathcal{C}_i}T_{\mathcal{B}} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

(ii) $${}_{\mathcal{B}}T_{\mathcal{C}_{ii}} = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} \qquad {}_{\mathcal{C}_{ii}}T_{\mathcal{B}} = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$$

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(b) (i)

$$[P_a]_{\mathcal{C}_i} = {}_{\mathcal{C}_i}T_{\mathcal{B}}\,[P_a]_{\mathcal{B}}\,{}_{\mathcal{B}}T_{\mathcal{C}_i} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$$

$$[P_b]_{\mathcal{C}_i} = {}_{\mathcal{C}_i}T_{\mathcal{B}}\,[P_b]_{\mathcal{B}}\,{}_{\mathcal{B}}T_{\mathcal{C}_i} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

(ii)

$$[P_a]_{\mathcal{C}_{ii}} = {}_{\mathcal{C}_{ii}}T_{\mathcal{B}}\,[P_a]_{\mathcal{B}}\,{}_{\mathcal{B}}T_{\mathcal{C}_{ii}} = \frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 3\\ -2 & 3\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}5/2 & 1/2\\ 1/2 & 5/2\end{bmatrix}$$

$$[P_b]_{\mathcal{C}_{ii}} = {}_{\mathcal{C}_{ii}}T_{\mathcal{B}}\,[P_b]_{\mathcal{B}}\,{}_{\mathcal{B}}T_{\mathcal{C}_{ii}} = \frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}1 & -1\\ -1 & -1\end{bmatrix}$$

6. Let $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ be an orthonormal basis for $\mathbb{E}^3$. Calculate the determinant and trace for each of the following linear operators:

(a)
$$\begin{aligned} P_a\colon \mathbb{R}^2 &\to \mathbb{R}^2\\ \mathbf{e}_x &\mapsto \mathbf{e}_x + \mathbf{e}_y\\ \mathbf{e}_y &\mapsto \mathbf{e}_y + \mathbf{e}_z\\ \mathbf{e}_z &\mapsto \mathbf{e}_x + \mathbf{e}_z \end{aligned}$$

(b)
$$\begin{aligned} P_b\colon \mathbb{R}^2 &\to \mathbb{R}^2\\ \mathbf{e}_x &\mapsto \mathbf{e}_x - \mathbf{e}_y\\ \mathbf{e}_y &\mapsto \mathbf{e}_y - \mathbf{e}_z\\ \mathbf{e}_z &\mapsto \mathbf{e}_x - \mathbf{e}_z \end{aligned}$$

(c)
$$\begin{aligned} P_c\colon \mathbb{R}^2 &\to \mathbb{R}^2\\ \mathbf{e}_x &\mapsto \mathbf{e}_x\\ \mathbf{e}_y &\mapsto -\mathbf{e}_z\\ \mathbf{e}_z &\mapsto -\mathbf{e}_y \end{aligned}$$

(d)
$$\begin{aligned} P_d\colon \mathbb{R}^2 &\to \mathbb{R}^2\\ \mathbf{e}_x &\mapsto \sqrt{2}\mathbf{e}_x - \mathbf{e}_y + \mathbf{e}_z\\ \mathbf{e}_y &\mapsto \sqrt{2}(\mathbf{e}_y + \mathbf{e}_z)\\ \mathbf{e}_z &\mapsto \sqrt{2}\mathbf{e}_x + \mathbf{e}_y - \mathbf{e}_z \end{aligned}$$

Solution:

(a)

$$[P_a]_{\mathcal{B}} = \begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix} \qquad \mathrm{Tr}(P) = 3 \qquad \det(P) = \det\left[\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right] + \det\left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right] = 2$$

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(b)

$$[P_b]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \qquad \mathrm{Tr}(P) = 1$$

For the determinant we can notice that the sum of the rows is zero, and hence the matrix is degenerate and $\det P = 0$. Alternatively we can calculate the determinant directly: $\det(P) = \det\left[\begin{smallmatrix} 1 & 0 \\ -1 & -1 \end{smallmatrix}\right] + \det\left[\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right] = 0$

(c)

$$[P_c]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \qquad \mathrm{Tr}(P) = 1 \qquad \det(P) = \det\left[\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right] = -1$$

(d)

$$[P_d]_{\mathcal{B}} = \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} \\ -1 & -\sqrt{2} & 1 \\ 1 & -\sqrt{2} & -1 \end{bmatrix} \qquad \mathrm{Tr}(P) = -1$$

$$\det(P) = \sqrt{2}\det\left[\begin{smallmatrix} -\sqrt{2} & 1 \\ -\sqrt{2} & -1 \end{smallmatrix}\right] + \sqrt{2}\det\left[\begin{smallmatrix} -1 & -\sqrt{2} \\ 1 & -\sqrt{2} \end{smallmatrix}\right] = \sqrt{2}(2\sqrt{2}) + \sqrt{2}(2\sqrt{2}) = 8$$

7. For each linear operator in question 6:

   (i) Determine if the operator is orthogonal or not.

   Note: *It may be useful to recall a fact we learn about the determinant of an orthogonal operator.*

   (ii) For $\lambda \in \mathbb{R}$ define the scaled linear operator $S\colon \mathbb{E}^3 \to \mathbb{E}^3$, by $S(\mathbf{v}) = \lambda P(\mathbf{v})$. Determine the values of λ (if any exist) such that S is an orthogonal operator.

   Solution:

   (i) The determinant of an orthogonal operator is either 1 or -1, so we can see immediately that P_a, P_b and P_d are **not** orthogonal operators.

   For P_c we can check if the matrix of the linear operator in the orthonormal basis

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$\mathcal{B}$, is an orthogonal matrix.

$$[P_c]_{\mathcal{B}}[P_c]_{\mathcal{B}}{}^{\mathsf{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} = I_3.$$

Since the product gives the identity, P_c is an orthogonal operator.

(ii) If S_a is orthogonal then the inner product $\langle S_a(\mathbf{e}_x), S_a(\mathbf{e}_y)\rangle = 0$, but

$$\begin{aligned}\langle S_a(\mathbf{e}_x), S_a(\mathbf{e}_y)\rangle &= \langle \lambda P_a(\mathbf{e}_x), \lambda P_a(\mathbf{e}_y)\rangle \\ &= \lambda^2 \langle \mathbf{e}_x + \mathbf{e}_y, \mathbf{e}_y + \mathbf{e}_z\rangle = \lambda^2.\end{aligned}$$

Thus if $\langle S_a(\mathbf{e}_x), S_a(\mathbf{e}_y)\rangle = 0$, we would necessarily require $\lambda = 0$. Since the zero linear operator is never orthogonal we see that S_a is not orthogonal for any choice of λ.

The operator P_b is degenerate, thus the scaled version S_b is also degenerate. It is therefore not orthogonal for any $\lambda \in \mathbb{R}$.

The operator P_c is orthogonal. It is easy to see that $-P_c$ is also orthogonal, but for any other choice of λ the determinant would not be ± 1. Thus S_c is orthogonal for $\lambda \in \{1, -1\}$.

The operator P_d has determinant 8 and the determinant of $S_d = \lambda^3 \det P_d$, thus the only possible choices for λ are $\pm 1/2$. Setting $\lambda = 1/2$ we see that

$$[S_d]_{\mathcal{B}}[S_d]_{\mathcal{B}}{}^{\mathsf{T}} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/2 & -1/\sqrt{2} & 1/2 \\ 1/2 & -1/\sqrt{2} & -1/2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/2 & 1/2 \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/2 & -1/2 \end{bmatrix} = I_3.$$

Similarly for $\lambda = -1/2$, so S_d is orthogonal for $\lambda \in \{1/2, -1/2\}$.

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8. (a) Group the following bases into equivalent classes with respect to orientation. That is, all bases in the same class must share an orientation and any pair in different classes must have a different orientation.

$$\mathcal{B}_1 = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) \qquad \mathcal{B}_2 = (\mathbf{e}_y, \mathbf{e}_z, \mathbf{e}_x) \qquad \mathcal{B}_3 = (\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z)$$
$$\mathcal{B}_4 = (\mathbf{e}_x + \mathbf{e}_y, \mathbf{e}_y - \mathbf{e}_x, \mathbf{e}_z) \quad \mathcal{B}_5 = (\mathbf{e}_y + \mathbf{e}_z, \mathbf{e}_z - \mathbf{e}_y, \mathbf{e}_x) \quad \mathcal{B}_6 = (\mathbf{e}_x + \mathbf{e}_y, \mathbf{e}_x - \mathbf{e}_y, \mathbf{e}_z)$$

(b) Using your answer to part (a), or otherwise, determine which of the following linear operators preserve the orientation and which change the orientation.

$$\begin{array}{llll} P_1 \colon \mathbb{E}^3 \to \mathbb{E}^3 & P_1(\mathbf{e}_x) = \mathbf{e}_y, & P_1(\mathbf{e}_y) = \mathbf{e}_z, & P_3(\mathbf{e}_z) = \mathbf{e}_x \\ P_2 \colon \mathbb{E}^3 \to \mathbb{E}^3 & P_2(\mathbf{e}_x) = \mathbf{e}_y, & P_2(\mathbf{e}_y) = \mathbf{e}_x, & P_3(\mathbf{e}_z) = \mathbf{e}_z \\ P_3 \colon \mathbb{E}^3 \to \mathbb{E}^3 & P_3(\mathbf{e}_x) = \mathbf{e}_x + \mathbf{e}_y, & P_3(\mathbf{e}_y) = \mathbf{e}_y - \mathbf{e}_x, & P_3(\mathbf{e}_z) = \mathbf{e}_z \\ P_4 \colon \mathbb{E}^3 \to \mathbb{E}^3 & P_4(\mathbf{e}_x) = \mathbf{e}_y + \mathbf{e}_z, & P_4(\mathbf{e}_y) = \mathbf{e}_z - \mathbf{e}_y, & P_3(\mathbf{e}_z) = \mathbf{e}_x \\ P_5 \colon \mathbb{E}^3 \to \mathbb{E}^3 & P_5(\mathbf{e}_x) = \mathbf{e}_x + \mathbf{e}_y, & P_5(\mathbf{e}_y) = \mathbf{e}_x - \mathbf{e}_y, & P_3(\mathbf{e}_z) = \mathbf{e}_z \end{array}$$

Solution:

(a) ${}_{\mathcal{B}_1}T_{\mathcal{B}_2} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. This has determinant 1 and so $\mathcal{B}_1$ and $\mathcal{B}_2$ have the same orientation.

${}_{\mathcal{B}_1}T_{\mathcal{B}_3} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This has determinant -1 and so $\mathcal{B}_1$ and $\mathcal{B}_3$ have opposite orientation.

${}_{\mathcal{B}_1}T_{\mathcal{B}_4} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This has determinant 2 and so $\mathcal{B}_1$ and $\mathcal{B}_4$ have the same orientation.

For the remaining transition matrices we can calculate them in the same way or we can note that

$${}_{\mathcal{B}_1}T_{\mathcal{B}_4} = {}_{\mathcal{B}_2}T_{\mathcal{B}_5} = {}_{\mathcal{B}_3}T_{\mathcal{B}_6}.$$

Therefore $\mathcal{B}_2$ and $\mathcal{B}_5$ have the same orientation as each other and similarly, $\mathcal{B}_3$ and $\mathcal{B}_6$ share an orientation.

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Putting this together we find that the equivalence classes for these bases are

$$\{\, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_4, \mathcal{B}_5 \,\} \qquad \text{and} \qquad \{\, \mathcal{B}_3, \mathcal{B}_6 \,\}.$$

(b) Notice the P_1 maps basis $\mathcal{B}_1$ to basis $\mathcal{B}_2$, thus the matrix of the linear operator $[P_1]_{\mathcal{B}_1}$ is the same as the transition matrix ${}_{\mathcal{B}_1}T_{\mathcal{B}_2}$. Using the determinant from part (a) we see that P_1 preserves orientation.

Similarly, the matrix of each linear operator P_i in the basis $\mathcal{B}_1$ is the same as the transition matrix ${}_{\mathcal{B}_1}T_{\mathcal{B}_{i+1}}$. Again using the determinants from part (a) we see that P_2 and P_5 change the orientation, whilst P_1, P_3 and P_4 preserve the orientation.

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