LECTURE 11: DESINGULARISATION AND BLOWING UP AT A POINT

In this lecture, we will describe certain special types of morphism of varieties, and use these to state the *Hironaka desingularisation theorem*. We will then go on to discuss one of the key ingredients of the proof of this theorem: how to blow up $\mathbb{C}^2$, and curves in $\mathbb{C}^2$, at a point.

1. The Hironaka desingularisation theorem

Note. The material in this section is not examinable, as it cannot be presented rigorously in our current setup of affine algebraic varieties; it is here as motivation for the development of blowups.

Let $F : V \to W$ be a morphism of varieties.

Definition 1 (Projective morphism)**.** *We say that F is a* projective morphism *if V is a subvariety of some product variety*

$$W \times \mathbb{P}^n$$

and F is the restriction to V of projection onto the first coordinate.

Note. Don't confuse *projective morphism* with *morphism of projective varieties*!

Examples.

(1) **Projection of $\mathbb{C}^2$ onto $\mathbb{C}$:** Let $V = \mathbb{C}^2$ and $W = \mathbb{C}$. Then V is a subvariety of $W \times \mathbb{P}^1 = \mathbb{C} \times \mathbb{P}^1$, and the map $F(x, y) = x$ is a projective morphism.

(2) **Projection of a parabola on $\mathbb{C}$:** Let $V = \mathbb{V}(Y - X^2) \subseteq \mathbb{C}^2$ and $W = \mathbb{C}$. Then the restriction of F in (1) to V is a projective morphism $V \to W$.

(3) **Projections of the twisted cubic:** Let $V = \mathbb{V}(Y - X^2, Z - X^3) \subseteq \mathbb{C}^3$ be the twisted cubic, and let $W_1 = \mathbb{V}(Y - X^2) \subseteq \mathbb{C}^2$ and $W_2 = \mathbb{V}(Z - X^3) \subseteq \mathbb{C}^2$ be the parabola and the cubic respectively. Then V is a subvariety of both $W_1 \times \mathbb{P}^1$ and $W_2 \times \mathbb{P}^1$, and the restrictions of $F_1(X, Y, Z) = (X, Y)$ and $F_2(X, Y, Z) = (X, Z)$ to V are projective morphisms $V \to W_1$ and $V \to W_2$ respectively.

Definition 2 (Birational morphism)**.** *We say that F is* birational *if there exist subvarieties*

$$V' \subset V \quad \textit{and} \quad W' \subset W$$

with

$$\dim V' < \dim V \quad \textit{and} \quad \dim W' < \dim W$$

such that F is an isomorphism

$$V \backslash V' \to W \backslash W'.$$

Note. All isomorphisms are trivially birational, but a birational morphism doesn't need to be injective or surjective.

Theorem 3 (Hironaka desingularisation theorem)**.** *Let V be a variety. Then then there exists a smooth variety D and a projective birational morphism*

$$\pi : D \to V$$

which is an isomorphism on the smooth locus of V.

What does this mean? If V is a variety, then no matter how many singularities it has, or how extreme these singularities are, it admits a *desingularisation* D, which is a variety which projects down to V and looks like V everywhere except at the singularities.

Examples of singular curves.

- Nodal curve: $\mathbb{V}(Y^2 - X^2(X+1))$
- Cone: $\mathbb{V}(X^2 + Y^2 - Z^2)$
- Cuspidal curve: $\mathbb{V}(Y^2 - X^3)$
- Whitney umbrella: $\mathbb{V}(X^2 - Y^2 Z)$

How do we get these desingularisations? One way is via *blowups*.

2. Blowing up

Ignore singularities for the moment, and let's just try to define the blowup of the origin in $\mathbb{C}^2$. What we want is to leave $\mathbb{C}^2$ unaltered everywhere except at 0, which we replace by the set of all lines passing through 0 (that is, a copy of $\mathbb{P}^1$).

Let B be given by

$$B = \{(x, \ell) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid x \in \ell\} \subset \mathbb{C}^2 \times \mathbb{P}^1.$$

The blowup of $\mathbb{C}^2$ at 0 is defined to be this set, along with the natural projection from B to its affine factor:

$$\begin{array}{rrcl} \pi : & B & \longrightarrow & \mathbb{C}^2 \\ & (x,\ell) & \longmapsto & x. \end{array}$$

Does this do what we want?

- What is the fiber $\pi^{-1}(x)$ over a point x which is not the origin? This will just be a single point (x, ℓ) of B, where ℓ is the unique line through x and the origin.
- What is the fibre of π over 0? This will be an entire copy of $\mathbb{P}^1$ (namely $\{0\} \times \mathbb{P}^1$).
- The map $\pi : B \to \mathbb{C}^2$ collapses $\{0\} \times \mathbb{P}^1$ to a point, and is bijective everywhere else.

A diagram of the blowup of the plane at a point was distributed in class, and can be found on p116 of Smith et al's book.

It can be shown that B is a *quasiprojective variety* (a more general type of variety which we have not studied in this course). Intersecting it with an affine chart of the projective line will give an affine variety.

The variety B, together with the projection map $\pi : B \to \mathbb{C}^2$, is sometimes referred to as the *one-point blowup of* $\mathbb{C}^2$, and denoted by $B_p(\mathbb{C}^2)$.

This process very easily generalises to a one-point blowup of $\mathbb{C}^n$, replacing $\mathbb{C}^2$ with $\mathbb{C}^n$ and $\mathbb{P}^1$ with $\mathbb{P}^{n-1}$. The resulting variety will be an n-dimensional quasiprojective variety embedding in $(2n-1)$-dimensional space.