LECTURE 12: BLOWING UP VARIETIES AT THE ORIGIN (THEORY AND EXAMPLES)

In this lecture, we will describe how to use our blow-up of $\mathbb{C}^n$ at the origin from last lecture to blow up subvarieties of $\mathbb{C}^n$ at the origin. We will describe the theory, and then give a selection of examples.

Our ultimate goal would be to blow up a given variety at an arbitrary subvariety (the origin, being just a point, is the most basic example of a subvariety).

1. Theory

Recall: we have the blowup of $\mathbb{C}^n$ at the origin, given by

$$B_0(\mathbb{C}^n) = \{(x,\ell) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid x \in \ell\}$$

and

$$\begin{array}{rccc} \pi: & B_0(\mathbb{C}^n) & \longrightarrow & \mathbb{C}^n \\ & (x,\ell) & \longmapsto & x. \end{array}$$

Definition 1. *Let $V \subset \mathbb{C}^n$ be a variety, and let $0 \in V$. Then the* blowup *of V at 0 is the closure of the preimage*

$$\pi^{-1}(V \backslash \{0\})$$

in $B_0(\mathbb{C}^n)$, together with the natural projection

$$\pi : \overline{\pi^{-1}(V \backslash \{0\})} \to V.$$

We denote the blowup of V at 0 by

$$B_0(V) := \overline{\pi^{-1}(V \backslash \{0\})}.$$

The restriction of π to $B_0(V) \backslash \pi^{-1}(0)$ is an isomorphism onto $V \backslash \{0\}$.

2. Examples of blowups

2.1. **Two lines crossing.**

We start with a very simple example. Consider the variety

$$V = \mathbb{V}(X^2 - Y^2) \subseteq \mathbb{C}^2.$$

This variety has a singularity at the origin, and hence is a good candidate to be blown up there.

We have our blowup map

$$\begin{array}{rccc} \pi: & B_0(\mathbb{C}^2) = \{(\mathbf{x},\ell) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid \mathbf{x} \in \ell\} & \longrightarrow & \mathbb{C}^2 \\ & (\mathbf{x},\ell) & \longmapsto & \mathbf{x}. \end{array}$$

Hence we have

$$\pi^{-1}(V \backslash \{0\}) = \{((x,y),\ell) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid (x,y) \in \ell,\ x^2 - y^2 = 0,\ (x,y) \neq 0\}.$$

Using projective coordinates $\ell = [s:t]$:

$$\pi^{-1}(V \backslash \{0\}) = \{((x,y),[s:t]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid xt - ys = 0,\ x^2 - y^2 = 0,\ (x,y) \neq 0\}.$$

To understand this as an affine variety, we intersect with an affine chart. Suppose $s \neq 0$: then we have (after renormalising to $s = 1$) that $\pi^{-1}(V\backslash\{0\}) \cap \mathbb{A}_s$ is given by

$$\{((x, y), [1 : t]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid xt - y = 0,\ x^2 - y^2 = 0,\ (x, y) \neq 0\}.$$

Since t can now range over all of $\mathbb{C}$, this is isomorphic to

$$\{(x, y, t) \in \mathbb{C}^3 \mid xt - y = 0,\ x^2 - y^2 = 0,\ (x, y) \neq 0\}.$$

Using the first equation to give $y = xt$, we get

$$\{(x, xt, t) \in \mathbb{C}^3 \mid x^2 - x^2t^2 = 0,\ (x, t) \neq 0\}.$$

Since $x \neq 0$, we can divide the equation by x^2:

$$\{(x, xt, t) \in \mathbb{C}^3 \mid 1 - t^2 = 0,\ (x, t) \neq 0\}.$$

The solutions to $1 - t^2$ are ± 1: so this variety breaks down as

$$\{(x, x, 1) \in \mathbb{C}^3 \mid x \neq 0\} \cup \{(x, -x, -1) \in \mathbb{C}^3 \mid x \neq 0\}.$$

Taking the closure of this gives

$$\{(x, x, 1) \mid x \in \mathbb{C}\} \cup \{(x, -x, -1) \mid x \in \mathbb{C}\},$$

or

$$\mathbb{V}(Y - X, Z - 1) \cup \mathbb{V}(Y + X, Z + 1) \subseteq \mathbb{C}^3.$$

2.2. **Nodal cubic.** The nodal cubic is given by

$$V = \mathbb{V}(Y^2 - X^2(X + 1)).$$

As before, using the blowup of $\mathbb{C}^2$ at the origin, we get

$$\pi^{-1}(V\backslash\{0\}) = \{((x, y), \ell) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid (x, y) \in \ell,\ y^2 = x^2(x + 1),\ (x, y) \neq 0\}.$$

Using projective coordinates $\ell = [s : t]$:

$$\pi^{-1}(V\backslash\{0\}) = \{((x, y), [s : t]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid xt - ys = 0,\ y^2 = x^2(x+1),\ (x, y) \neq 0\}.$$

Intersect with the affine chart where $s \neq 0$ (we choose our chart to give $y = xt$, since y is of lower degree than x in the equation): we get

$$\{((x, y), [1 : t]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid xt - y = 0,\ y^2 = x^2(x + 1),\ (x, y) \neq 0\}.$$

This is isomorphic to

$$\{(x, y, t) \in \mathbb{C}^3 \mid y = xt,\ y^2 = x^2(x + 1),\ (x, y) \neq 0\}.$$

This gives

$$\{(x, xt, t) \in \mathbb{C}^3 \mid x^2t^2 = x^2(x + 1),\ (x, t) \neq 0\}.$$

Dividing through by x^2 in the equation (since $x \neq 0$) gives

$$\{(x, xt, t) \in \mathbb{C}^3 \mid t^2 = x + 1,\ (x, t) \neq 0\},$$

and hence

$$\{(t^2 - 1, t^3 - t, t) \mid t \in \mathbb{C}, t \notin \{-1, 0, 1\}.$$

(We get $t \neq \pm 1$ since $x \neq 0$, giving $t^2 \neq 1$.) Closing this gives

$$\{(t^2 - 1, t^3 - t, t) \mid t \in \mathbb{C}\}.$$

2.3. **Cuspidal cubic.** The cuspidal cubic is given by

$$\mathbb{V}(Y^2 - X^3) \subseteq \mathbb{C}^2.$$

As before, using the blowup of $\mathbb{C}^2$ at the origin, we get

$$\pi^{-1}(V\backslash\{0\}) = \{((x,y),\ell) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid (x,y) \in \ell,\ y^2 = x^3,\ (x,y) \neq 0\}.$$

Using projective coordinates $\ell = [s:t]$:

$$\pi^{-1}(V\backslash\{0\}) = \{((x,y),[s:t]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid xt - ys = 0,\ y^2 = x^3,\ (x,y) \neq 0\}.$$

Intersect with the affine chart where $s \neq 0$ (we choose our chart to give $y = xt$, since y is of lower degree than x in the equation): we get

$$\{((x,y),[1:t]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid xt - y = 0,\ y^2 = x^3,\ (x,y) \neq 0\}.$$

This is isomorphic to

$$\{(x,y,t) \in \mathbb{C}^3 \mid y = xt,\ y^2 = x^3,\ (x,y) \neq 0\}.$$

This gives

$$\{(x,xt,t) \in \mathbb{C}^3 \mid x^2t^2 = x^3,\ (x,t) \neq 0\}.$$

Dividing through by x^2 in the equation (since $x \neq 0$) gives

$$\{(x,xt,t) \in \mathbb{C}^3 \mid t^2 = x,\ (x,t) \neq 0\},$$

and hence

$$\{(t^2,t^3,t) \mid t \in \mathbb{C}, t \neq 0\}.$$

Closing this gives

$$\{(t^2,t^3,t) \mid t \in \mathbb{C}\},$$

which is the twisted cubic. So the blowup of the cuspidal cubic is the twisted cubic (which we know is isomorphic to $\mathbb{C}$: so the cuspidal cubic and the complex line are birational).

2.4. **Bowtie curve.** The bowtie curve is given by

$$\mathbb{V}(X^4 + Y^3 - X^2Y) \subseteq \mathbb{C}^2.$$

This is singular at the origin. As before, using the blowup of $\mathbb{C}^2$ at the origin, we get

$$\pi^{-1}(V\backslash\{0\}) = \{((x,y),\ell) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid (x,y) \in \ell,\ x^2y = x^4 + y^3,\ (x,y) \neq 0\}.$$

Using projective coordinates $\ell = [s:t]$:

$$\pi^{-1}(V\backslash\{0\}) = \{((x,y),[s:t]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid xt - ys = 0,\ x^2y = x^4 + y^3,\ (x,y) \neq 0\}.$$

Intersect with the affine chart where $s \neq 0$ (we choose our chart to give $y = xt$, since y is of lower degree than x in the equation): we get

$$\{((x,y),[1:t]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid xt - y = 0,\ x^2y = x^4 + y^3,\ (x,y) \neq 0\}.$$

This is isomorphic to

$$\{(x,y,t) \in \mathbb{C}^3 \mid y = xt,\ x^2y = x^4 + y^3,\ (x,y) \neq 0\}.$$

This gives

$$\{(x,xt,t) \in \mathbb{C}^3 \mid x^3t = x^4 + x^3t^3,\ (x,t) \neq 0\}.$$

Dividing through by x^3 in the equation (since $x \neq 0$) gives

$$\{(x,xt,t) \in \mathbb{C}^3 \mid t = x + t^3,\ (x,t) \neq 0\},$$

and hence (since $x = t - t^3$) we have

$$\{(t - t^3, t^2 - t^4, t) \mid t \in \mathbb{C}, t \notin \{-1, 0, 1\}\}.$$

Closing this gives

$$\{(t - t^3, t^2 - t^4, t) \mid t \in \mathbb{C}\}.$$

2.5. **Cone.** Let's try to explicitly blow up the cone

$$V = \mathbb{V}(x^2 + y^2 - z^2) \subset \mathbb{C}^3$$

at the origin.

Informally: the cone looks like a union of lines, tied together at the origin; tying together these lines at the origin causes the cone to have a singularity there. We want to desingularise the cone, and we do so by separating these lines, taking their *disjoint* union and hence forming a cylinder.

Explicitly: consider the blowup map

$$\begin{array}{rrcl} \pi: & B_0(\mathbb{C}^3) = \{(x, \ell) \in \mathbb{C}^3 \times \mathbb{P}^2 \mid x \in \ell\} & \longrightarrow & \mathbb{C}^3 \\ & (x, \ell) & \longmapsto & x. \end{array}$$

If we look at $\pi^{-1}(V \backslash \{0\})$, we get

$$\{((x, y, z), \ell) \in \mathbb{C}^3 \times \mathbb{P}^2 \mid x \in \ell,\ x^2 + y^2 = z^2,\ \mathbf{x} \neq 0\}.$$

Using projective coordinates: we get

$$\left\{ ((x, y, z), [t : u : v]) \;\middle|\; \begin{array}{c} x^2 + y^2 = z^2 \\ xu - yt = 0 \\ yv - zu = 0 \\ xv - zt = 0 \end{array} \right\}.$$

Intersect with the chart where $v \neq 0$: so we have

$$\left\{ ((x, y, z), [t : u : 1]) \;\middle|\; \begin{array}{c} x^2 + y^2 = z^2 \\ xu - yt = 0 \\ y - zu = 0 \\ x - zt = 0 \end{array} \right\}.$$

This is isomorphic to

$$\left\{ (x, y, z, t, u) \in \mathbb{C}^5 \;\middle|\; \begin{array}{c} x^2 + y^2 = z^2 \\ xu - yt = 0 \\ y - zu = 0 \\ x - zt = 0 \end{array} \right\}.$$

Using the second two equations:

$$\left\{ (zt, zu, z, t, u) \in \mathbb{C}^5 \;\middle|\; z^2t^2 + z^2u^2 = z^2 \right\}.$$

Since $z \neq 0$, we can divide through by z^2:

$$\left\{ (zt, zu, z, t, u) \in \mathbb{C}^5 \;\middle|\; t^2 + u^2 = 1 \right\}.$$

We can project this into $\mathbb{C}^3$: we get the cylinder

$$\{(z, t, u) \mid t^2 + u^2 = 1, z \neq 0\}.$$

The preimage of $(0, 0, 0) \in \mathbb{V}(x^2 + y^2 - z^2)$ is the circle

$$\{(0, t, u) \mid t^2 + u^2 = 1\}$$

on the cylinder.

2.6. **Whitney umbrella.** The Whitney umbrella is given by

$$V = \mathbb{V}(X^2 - ZY^2).$$

This has a singularity at the origin (although this is not the only place it is singular!) Consider the blowup map

$$\begin{array}{rrcl} \pi: & B_0(\mathbb{C}^3) = \{(x,\ell) \in \mathbb{C}^3 \times \mathbb{P}^2 \mid x \in \ell\} & \longrightarrow & \mathbb{C}^3 \\ & (x,\ell) & \longmapsto & x. \end{array}$$

If we look at $\pi^{-1}(V \backslash \{0\})$, we get

$$\{((x,y,z),\ell) \in \mathbb{C}^3 \times \mathbb{P}^2 \mid x \in \ell,\ x^2 = zy^2,\ \mathbf{x} \neq 0\}.$$

Using projective coordinates: we get

$$\left\{ ((x,y,z),[t:u:v]) \;\middle|\; \begin{array}{c} x^2 = zy^2 \\ xu - yt = 0 \\ yv - zu = 0 \\ xv - zt = 0 \end{array} \right\}.$$

Intersect with the chart where $v \neq 0$: so we have

$$\left\{ ((x,y,z),[t:u:1]) \;\middle|\; \begin{array}{c} x^2 = zy^2 \\ xu - yt = 0 \\ y - zu = 0 \\ x - zt = 0 \end{array} \right\}.$$

This is isomorphic to

$$\left\{ (x,y,z,t,u) \;\middle|\; \begin{array}{c} x^2 = zy^2 \\ xu = yt \\ y = zu \\ x = zt \end{array} \right\}.$$

Using the third and fourth equations:

$$\left\{ (zt,zu,z,t,u) \in \mathbb{C}^5 \;\middle|\; u^2 = zt^2 \right\}.$$

Since $z \neq 0$, we can divide through by z^2:

$$\left\{ (zt,zu,z,t,u) \in \mathbb{C}^5 \;\middle|\; u^2 = zt^2 \right\}.$$

But this has exactly the same singularity as the original Whitney umbrella! So we need better blowup methods. There is no scope for this in this course: but what we would need to do in this case, for example, is to blow up the whole z-axis, and *then* blow up at the origin.