LECTURE 3: IRREDUCIBLE VARIETIES AND MORPHISMS OF VARIETIES

In this lecture, we will give definitions of *irreducible* and *reducible* varieties, of the *dimension* of a variety, and of a *morphism* of varieties, and give some examples.

1. Irreducible varieties

Definition 1. *We say a variety V is* reducible *if we can write V as the non-trivial union of two varieties. That is, we can write*

$$V = V_1 \cup V_2,$$

where $V_1 \neq V$ and $V_2 \neq V$ are varieties.

If V is not reducible, we say that V is irreducible*.*

When does this happen in the case of curves in $\mathbb{C}^2$? Recall from last lecture that we have

$$\mathbb{V}(F) \cup \mathbb{V}(G) = \mathbb{V}(F \cdot G).$$

So a variety $V = \mathbb{V}(F)$ defined by one polynomial will be irreducible if and only if F is a power of an irreducible polynomial g. (Recall: a polynomial G is irreducible if it cannot be written as $G = G_1G_2$ for two non-constant polynomials G_1, G_2.)

Examples of irreducible varieties.

(1) The parabola $\mathbb{V}(Y - X^2) \subseteq \mathbb{C}^2$ is irreducible.
(2) The circle $\mathbb{V}(X^2 + Y^2 - 1) \subseteq \mathbb{C}^2$ is irreducible.
(3) The elliptic curve $\mathbb{V}(Y^2 - X^3 + X - 1) \subseteq \mathbb{C}^2$ is irreducible.
(4) A line (e.g. $\mathbb{V}(Y - X) \subseteq \mathbb{C}^2$) is irreducible.
(5) The cuspical cubic $\mathbb{V}(Y^2 - X^3) \subseteq \mathbb{C}^2$ is irreducible.
(6) The *twisted cubic*, defined by $\mathbb{V}(X^2 - Y, X^3 - Z) \subseteq \mathbb{C}^3$, is irreducible.

Examples of reducible varieties.

(1) The variety $\mathbb{V}(X^2 - Y^2)$ is not irreducible, as we have $X^2 - Y^2 = (X + Y)(X - Y)$, and hence

$$\mathbb{V}(X^2 - Y^2) = \mathbb{V}(X + Y) \cup \mathbb{V}(X - Y).$$

(2) Despite first appearances, the variety $\mathbb{V}(X^2 + Y^2)$ is not irreducible, as we are working over $\mathbb{C}$ and hence we have $X^2 + Y^2 = (X + iY)(X - iY)$.
(3) The variety

$$\mathbb{V}(X^2 - YZ, XZ - X)$$

from Problems Class 1 is reducible. Recall that we divided it up into the two lines $X = Y = 0$ and $X = Z = 0$ and the parabola $Z - 1 = Y - X^2 = 0$: so we have

$$\mathbb{V}(X^2 - YZ, XZ - X) = \mathbb{V}(X, Y) \cup \mathbb{V}(X, Z) \cup \mathbb{V}(Z - 1, Y - X^2).$$

(4) Note that the intersection of two irreducible varieties is not necessarily irreducible: consider

$$\mathbb{V}(Y - X^2, 1 - Y - X^2) = \mathbb{V}(Y - X^2) \cap \mathbb{V}(1 - Y - X^2).$$

Both $\mathbb{V}(Y - X^2)$ and $\mathbb{V}(1 - Y - X^2)$ are irreducible varieties, but their intersection is the set of two points

$$\left\{ \left(\frac{\sqrt{2}}{2}, \frac{1}{2} \right), \left(-\frac{\sqrt{2}}{2}, \frac{1}{2} \right) \right\}.$$

As mentioned last lecture, a single point is always a variety. So this variety is the non-trivial union of two varieties, and hence not irreducible.

2. The dimension of a variety

Definition 2. *Let V be a variety. Then a* subvariety W *of V is just a subset of V which is itself a variety.*

Now that we have the notions of an irreducible variety and a subvariety, we can give the definition of the *dimension* of a variety.

Definition 3. *Let V be a variety. Then the* dimension $\dim V$ *of V is the length of the longest possible chain*

$$V_d \supsetneq V_{d-1} \supsetneq \cdots \supsetneq V_1 \supsetneq V_0,$$

where the V_i are distinct non-empty irreducible subvarieties of V.

We can also define the dimension $\dim_x(V)$ of V near a point $x \in V$. This is defined to be the length of the longest possible chain of distinct non-empty irreducible subvarieties of V *ending in* $\{x\}$: i.e,

$$V_d \supsetneq V_{d-1} \supsetneq \cdots \supsetneq V_1 \supsetneq \{x\}.$$

Most varieties have the dimension one would expect: however, it is often very difficult to actually prove that a given variety has the dimension one would expect. Even showing that $\mathbb{C}^n$ is n-dimensional is non-trivial! This is an area of the subject which benefits greatly from highly algebraic machinery.

3. Morphisms of varieties

Definition 4. *Let $V \subseteq \mathbb{C}^n$ and $W \subseteq \mathbb{C}^m$ be varieties. A map $F : V \to W$ is a* morphism of varieties *if it is the restriction of a polynomial map $\mathbb{C}^n \to \mathbb{C}^m$. We call a morphism of varieties an* isomorphism *if it is bijective and if its inverse is also a morphism.*

Examples.

(1) The projection map

$$\pi_X : \begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & \mathbb{C} \\ (x, y) & \longmapsto & x \end{array}$$

is a morphism from $\mathbb{C}^2$ to $\mathbb{C}$. It is not an isomorphism, as it is not injective.

(2a) Let $V = \mathbb{V}(Y - X^2)$. Then we have a bijective morphism $\mathbb{C} \to V$ given by

$$t \mapsto (t, t^2).$$

It has an inverse $V \to \mathbb{C}$ given by (the restriction of) the projection map $(x, y) \mapsto x$, and hence this morphism is an isomorphism.

(2b) Consider the restriction of the *other* projection map $V \to \mathbb{C}$ given by $(x, y) \mapsto y$. This morphism is surjective, but not injective, and hence does not have an inverse.

(3) Let

$$M = \begin{pmatrix} m_{11} & \cdots & m_{1,n} \\ \vdots & \cdots & \vdots \\ m_{n,1} & \cdots & m_{n,n} \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}).$$

Then the map

$$\begin{array}{rccc} f_M : & \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ & \mathbf{z} & \longmapsto & M\mathbf{z} \end{array}$$

is an isomorphism from $\mathbb{C}^n$ to $\mathbb{C}^n$. (An isomorphism from a space to itself is usually called an *automorphism.*)

It should be noted that a morphism between two algebraic varieties *doesn't* necessarily send subvarieties to subvarieties. Consider again the projection map

$$\begin{array}{rccc} \pi_X : & \mathbb{C}^2 & \longrightarrow & \mathbb{C} \\ & (x,y) & \longmapsto & x. \end{array}$$

Now consider the subvariety $H = \mathbb{V}(XY-1)$ of $\mathbb{C}^2$. This is a hyperbola, and can be given by

$$H = \{(t, t^{-1}) \mid t \in \mathbb{C}, t \neq 0\}.$$

The image of H under π_X is just $\mathbb{C}\backslash\{0\}$, which is *not* a subvariety of $\mathbb{C}$ (it is open in the Euclidean topology). So while π_X is a morphism $\mathbb{C}^2 \to \mathbb{C}$ of the ambient varieties, it can send subvarieties to sets which are no longer varieties.

Note. Under a more general definition of "variety", this set would be considered to be a variety.

If there exists an isomorphism f between two varieties V and W, we say that V and W are isomorphic.

Example. The twisted cubic

$$V = \mathbb{V}(X^2 - Y, X^3 - Z)$$

is isomorphic to $\mathbb{C}$. To see this, we first show that V can be written as

$$V = \{(t, t^2, t^3) \mid t \in \mathbb{C}\}.$$

Then there exists a clear map $f : \mathbb{C} \to V$ sending t to (t, t^2, t^3), and its inverse is projection to the first coordinate.

Next lecture: we will move on to a more abstract part of the course. We will recall some definitions from abstract algebra (prime, maximal and radical ideals of a ring), define *Noetherian rings* and explore their properties, and define the *variety of an ideal.*