LECTURE 4: NOETHERIAN RINGS AND THE IDEAL-VARIETY CORRESPONDENCE

In this lecture, we will recall the definitions of various types of ideal (*prime*, *maximal* and *radical* ideals), define a *Noetherian ring* and look at certain properties of these rings, and define the *variety of an ideal* and the *ideal of a variety.*

1. Review of ideals

In this course, all rings are commutative and have a multiplicative identity element.

Recall the following definitions:

Definition 1. *Let R be a ring. We say that $I \subseteq R$ is an* ideal *of R if*

(i) *I is an additive subgroup of R;*
(ii) *If $i \in I$ and $r \in R$, then $ir \in I$.*

There are certain types of ideals which will be of particular interest:

Definition 2. *An ideal $\mathfrak{p} \subsetneq R$ is* prime *if whenever we have $fg \in \mathfrak{p}$, we must have either $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.*

Definition 3. *An ideal $\mathfrak{m} \subsetneq R$ is* maximal *if whenever we have $\mathfrak{m} \subsetneq I$ for some ideal I, we must have $I = R$.*

Definition 4. *The* radical *of an ideal $I \subset R$ is defined to be the ideal*

$$\sqrt{I} := \{f \in R \mid f^n \in I \text{ for some } n \in \mathbb{N}\}.$$

An ideal is said to be radical *if it is equal to its radical, i.e. if $I = \sqrt{I}$.*

Let $X \subseteq R$. Then we can define the ideal of R generated by X to be the ideal

$$\{x_1 r_1 + \cdots + x_n r_n \mid x_i \in X, r_i \in R, n \in \mathbb{N}\}.$$

If X is a finite set $\{x_1, \ldots, x_n\}$, then we write

$$(x_1, \ldots, x_n) = \{x_1 r_1 + \cdots + x_n r_n \mid x_i \in X, r_i \in R\}.$$

Examples:

(1) Consider the ring $\mathbb{Z}$.
   - (2) is an ideal of $\mathbb{Z}$. It is prime, maximal and radical.
   - (27) is an ideal of $\mathbb{Z}$. It is neither prime, maximal nor radical: its radical is (3).
   - (0) is an ideal of $\mathbb{Z}$. It is prime and radical, but not maximal.

(2) Consider the ring $\mathbb{Z}/16\mathbb{Z}$. Then the zero ideal is not prime, radical or maximal: its radical is (2).

(3) Consider the ring $\mathbb{Z}[X]$.
   - (0), (p) and (X) are prime and radical, but not maximal.
   - (p, X) is prime, radical and maximal.

2. Noetherian rings

Definition 5. *We say that a ring R is a* Noetherian ring *if for every ascending chain of ideals*

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots,$$

there is an integer N such that $I_N = I_{N+m}$ for all $m \in \mathbb{N}$. That is, every strictly ascending chain of ideals must terminate.

We call this property the *Noetherian property* or the *ascending chain condition.*

Proposition 6. *Let R be a ring. The following are equivalent:*

1. *R is a Noetherian ring;*
2. *Every non-empty set of ideals of R has a maximal element with respect to inclusion (i.e. an ideal which is not contained in any of the others).*
3. *Every ideal $I \subseteq R$ is finitely generated. That is, there are $f_1, \ldots, f_k \in I$ such that $I = (f_1, \ldots, f_k)$.*

Proof. (1) $\Rightarrow$ (2): Let $\mathcal{I}$ be a non-empty set of ideals, and let $I_1 \in \mathcal{I}$. If I_1 is maximal, then we are done; otherwise, there exist $I_2 \supsetneq I_1$. We can continue inductively to form an ascending chain of ideals, and this chain must terminate by (1), giving a maximal element.

(2) $\Rightarrow$ (3): Let I be an ideal of R. Define the set

$$\mathcal{I} = \big\{ J \subseteq I \mid J \text{ is a finitely generated ideal} \big\}$$

of finitely generated ideals contained in I. Let $J = (j_1, \ldots, j_n)$ be a maximal element of $\mathcal{I}$, and suppose that $J \neq I$. Then there must be some element $f \in I \backslash J$. But then we can form another finitely generated ideal $(j_1, \ldots, j_n, f) \supsetneq J$, which contradicts the maximality of J. So we must have $J = I$, and hence I is finitely generated.

(3) $\Rightarrow$ (1): Left as an exercise (Q5, HW 1). □

Example. The ring of integers $\mathbb{Z}$ is Noetherian. (This is left as an exercise: see Q6, HW 1.)

Proposition 7. *If R is a Noetherian ring and I is an ideal of R, then the quotient ring R/I is Noetherian.*

Proof. There is a one-to-one correspondence between ideals of R/I and ideals of R containing I. So let $J \supseteq I$ be an ideal of R such that J/I is the corresponding ideal of R/I.

Since we have $J = (f_1, \ldots, f_n)$ for some $f_i \in R$, we have

$$J/I = (f_1 + I, \ldots, f_m + I),$$

and hence J/I is finitely generated. □

Theorem 8 (Hilbert basis theorem)**.** *Let R be a Noetherian ring. Then the polynomial ring $R[X]$ is Noetherian.*

Proof. Let $I \subseteq R[X]$ be an ideal. We will prove that I is finitely generated. For each non-negative integer n define the leading term ideal of degree n to be the ideal

$$I_n = \{a_n \in R \mid \text{there exists some } f = a_n X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in I\}$$

As an exercise prove that this is indeed an ideal and that $I_n \subseteq I_{n+1}$ for each n. Hence these ideals form an ascending chain thus there is an N such that $I_N = I_{N+1} = \cdots$.

Since R is Noetherian each I_n is finitely generated, so we may pick generators $a_{n,1}, \ldots, a_{n,m_n}$. Now for each of these let $f_{n,k}$ be a degree n polynomial with leading term $a_{n,k}$, which must exist by the definition of I_n. We will prove that the set

$$\{f_{n,k} \mid 0 \leq n \leq N \text{ and } 1 \leq k \leq m_n\}$$

generates I.

Assume for a contradiction that the $f_{n,k}$ do not generate I. Let $g \in I$ be an element of minimal degree such that g is not generated by the $f_{n,k}$. Suppose $\deg g = n$ then the leading term of g is bX^n for some $b \in I_n$. Now let $n' = n$ if $n \leq N$ and $n' = N$ otherwise. We can write b in terms of the generators of $I_n = I_{n'}$:

$$b = \sum_{k=1}^{m_{n'}} c_k a_{n',k}$$

for $c_k \in R$. Now consider $h = g - X^{n-n'} \sum c_k f_{n',k}$ by construction $\deg h < \deg g$ and so by minimality of g, we must have that h is generated by the $f_{n,k}$, say $h = \sum d_{m,k} f_{m,k}$. Now

$$g = X^{n-n'} \sum c_k f_{n',k} + \sum d_{m,k} f_{m,k}$$

contradicting the assumption and therefore the $f_{m,k}$ do indeed generate I. □

Corollary 9. *If R is Noetherian, then $R[X_1, \ldots, X_n]$ is Noetherian.*

3. The variety of an ideal and ideal of a variety

In what follows, we will use k to denote a general field (sometimes we will specify that it is algebraically closed). However, there is no harm in just thinking of this as $\mathbb{C}$.

Let k be a field. Let $R = k[X_1, \ldots, X_n]$, and let $f \in R$. The polynomial f defines a map $f : k^n \to k$ by evaluating the polynomial at points $P = (x_1, \ldots, x_n)$ of k^n. We can use this to define a correspondence

$$\begin{array}{rcl} \{\text{ideals } J \subseteq R\} & \longrightarrow & \{\text{subsets } X \subseteq k^n\} \\ J & \longmapsto & \mathbb{V}(J) = \{P \in k^n \mid f(P) = 0 \text{ for all } f \in J\}. \end{array}$$

For an ideal J, we call $\mathbb{V}(J)$ the *variety of J*.

Proposition 10. *The correspondence $J \mapsto \mathbb{V}(J)$ satisfies the following properties:*

(1) $\mathbb{V}(0) = k^n$
(2) $\mathbb{V}(R) = \emptyset$
(3) $J_1 \subseteq J_2 \;\Rightarrow\; \mathbb{V}(J_1) \supseteq \mathbb{V}(J_2)$
(4) $\mathbb{V}(J_1 \cap J_2) = \mathbb{V}(J_1) \cup \mathbb{V}(J_2)$
(5)
$$\mathbb{V}\left(\sum_{\lambda \in \Lambda} J_\lambda\right) = \bigcap_{\lambda \in \Lambda} \mathbb{V}(J_\lambda).$$

Proof. Left as an exercise. □

We also have a partial inverse to this correspondence. Again, let k be a field and let $R = k[X_1, \ldots, X_n]$. Then we have a correspondence

$$\begin{array}{rcl} \{\text{subsets } X \subseteq k^n\} & \longrightarrow & \{\text{ideals } J \subseteq R\} \\ X & \longmapsto & \mathbb{I}(X) = \{f \in R \mid f(P) = 0 \text{ for all } P \in X\}. \end{array}$$

It should be clear that $\mathbb{I}(X)$ is an ideal.

Proposition 11. *Let k be a field, and let $R = k[X_1, \ldots, X_n]$.*

(1) $X \subseteq k^n \;\Rightarrow\; X \subseteq \mathbb{V}(\mathbb{I}(X))$ *with equality if and only if X is a variety;*
(2) $X \subseteq Y \subseteq k^n \mid \mathbb{I}(X) \supseteq \mathbb{I}(Y)$*;*
(3) $J \supseteq R \;\Rightarrow\; J \subseteq \mathbb{I}(\mathbb{V}(J))$. *This inclusion can be strict.*

Remark. Note that part (1) now confirms that all varieties are generated by finitely many polynomials. To see this, let V be a variety: by part (1) of the above proposition, we have

$$V = \mathbb{V}(\mathbb{I}(V)).$$

The ideal $\mathbb{I}(V)$ is an ideal in $k[X_1, \ldots, X_n]$. By the Hilbert basis theorem, $k[X_1, \ldots, X_n]$ is Noetherian (all fields are Noetherian), and hence the ideal $\mathbb{I}(V)$ is finitely generated. So we can write

$$\mathbb{I}(V) = (F_1, \ldots, F_m)$$

for some $F_i \in k[X_1, \ldots, X_n]$, and hence

$$V = \mathbb{V}(F_1, \ldots, F_m).$$