1 Tangent space and singularity

We will restrict our attention to complex varieties in this section.

Our aim is to generalise our understanding of tangency to general varieties. We understand that the x-axis is tangent to the curve $y = x^2$ at the origin because the curve has a multiple root at this point. Similarly we can understand the generalised idea in a similar algebraic manner.

Let us assume that a complex variety V contains the origin and let ℓ be a line through the origin. If we fix any point $P = (a_1, \ldots, a_n)$ on ℓ then the line may be parameterised by $\ell = \{(ta_1, \ldots, ta_n) \mid t \in \mathbb{C}\}$. We wish to understand when ℓ is tangent to V at the origin.

Let $\mathbb{I}(V) = (f_1, \ldots, f_r)$ then the points in $V \cap \ell$ are given by solving the system of equations

$$\begin{aligned} f_1(ta_1, \ldots, ta_n) &= 0 \\ &\vdots \\ f_r(ta_1, \ldots, ta_n) &= 0 \end{aligned}$$

for values of t. Each f_i is a polynomial in one variable t, and therefore separates into linear factors. The points in $V \cap \ell$ are given by the linear factors shared by all r equations. The multiplicity of a root is simply the number of times it is shared by all equations. In particular the multiplicity of the root at 0 is just given by the largest power of t dividing all equations.

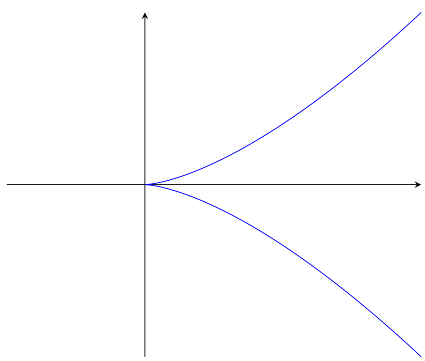
Note. It is important that the f_i generate the whole radical ideal $\mathbb{I}(V)$ and are not simply generators of the variety, as otherwise multiplicity will be ill-defined.

Definition 1 (Tangent)**.** The line ℓ is *tangent* to the variety V at the point P if the multiplicity of the of the root at P is greater than one. The point is *tangent of order n* if it has a root of multiplicity $n+1$ at that point.

The *tangent space $T_P V$* of V at the point P is the union of all lines tangent to V at P.

Example 2. Let V be the variety defined by the polynomial $y - x^2$. Let $\ell = \{(ta, tb) \mid t \in \mathbb{C}\}$. The intersection $V \cap \ell$ is given by solutions to $tb - t^2a^2 = 0$. That is the point $(0,0)$ corresponding to $t = 0$ and the point $\left(\frac{b}{a}, \frac{b^2}{a^2}\right)$ corresponding to $t = \frac{b}{a^2}$. The line ℓ is then only tangent if these two points coincide, that is if $b = 0$ and we retrieve learn again that the only tangent line is the x-axis.

Example 3. Let V be the variety defined by the polynomial $y^2 - x^3$. Now the intersection with the line ℓ is given by solutions to $t^2b^2 - t^3a^3 = 0$. Clearly for any values of (a, b) this has a multiple root at 0 and thus all lines are tangent to the origin, that is $T_0 V = \mathbb{C}^2$. We will see that this is an example of a *singular point*.



Just as we can understand tangency for simple functions in terms of derivatives, the same is true of varieties.

Definition 4 (Differential)**.** The differential $df|_0$ of a polynomial f at the origin is the linear part of f. That is, the sum of all degree 1 parts of f.

More generally, for a point $p = (a_1, \dots, a_n)$ we may consider f as a polynomial in the variables $x_1 - a_1, \dots, x_n - a_n$. This is simply the Taylor expansion of f at p. The differential $df|_p$ of f at p is now the linear part of this Taylor expansion.

Theorem 5:

Let $V \subseteq \mathbb{C}^n$ be a variety and let $\mathbb{I}(V) = (f_1, \dots, f_r)$.

The tangent space of V at the origin is the linear variety

$$T_0V = \mathbb{V}(df_1|_0, \dots, df_r|_0) \subseteq \mathbb{C}^n.$$

In particular the space in independent of the choice of generators.

By utilising the Taylor expansion we obtain an equivalent statement for the tangent space T_pV at an arbitrary point p.

Proof. Consider the line ℓ through the fixed point $(x_1, \dots, x_n)$. As the point 0 is assumed to be in V we know that the constant part of each f_i is zero. Therefore

$$f_i(tx_1, \dots, tx_n) = df_i|_0(tx_1, \dots, tx_n) + F_i(tx_1, \dots, tx_n)$$

where $df_i|_0(tx_1, \dots, tx_n) = tdf_i|_0(x_1, \dots, x_n)$ is homogeneous of degree 1 and $F_i(tx_1, \dots, tx_n)$ has all terms divisible by t^2. We wish to find the values of t for which the multiplicity of the root is at least two, clearly this can happen if and only if all the linear terms are zero

$$df_1|_0(x_1, \ldots, x_n) = 0$$
$$\vdots$$
$$df_r|_0(x_1, \ldots, x_n) = 0$$

We have shown that $T_0 V = \mathbb{V}(df_1|_0, \ldots, df_r|_0)$ as required.

We leave the proof that $T_0 V$ is independent of the choice of generators as an exercise. □

Definition 6. A point P on a variety V is a *smooth* point if $\dim T_P V = \dim_p V$. Otherwise, $p \in V$ is called a *singular* point.

The following theorem is a useful tool for calculating the set of singular points of a variety. We state this theorem without proof.

Theorem 7:

Let V be an irreducible variety in $\mathbb{C}^n$ of dimension d and let the radical ideal $\mathbb{I}(V) = (F_1, \ldots, F_r)$.

The singular locus of V is the set of common zeros in V of the polynomials obtained as determinants of $(n-d) \times (n-d)$–minors of the Jacobian matrix.

$$\begin{pmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_2}{\partial X_2} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_2}{\partial X_2} \end{pmatrix}$$

Example 8. We calculate the singular locus of $\mathbb{V}(Y^2 - X^3)$ as in example 3. The Jacobian matrix is given by

$$\begin{pmatrix} \frac{\partial F}{\partial X} & \frac{\partial F}{\partial Y} \end{pmatrix} = \begin{pmatrix} -3X^2 & 2Y \end{pmatrix}$$

Now $n - d = 1$ and so the determinants of the 1×1–minors are simply $-3X^2$ and $2Y$. Thus the singular locus is given by

$$\mathbb{V}(Y^2 - X^3) \cap \mathbb{V}(-3X^2) \cap \mathbb{V}(2Y)$$

and is the single point $(0, 0)$.