PROBLEMS CLASS — WEEK 1

Questions

(1) Show that the given sets are varieties, and write them in the standard notation $\mathbb{V}(\{F_i\}_{i\in I}) \subset \mathbb{C}^n$.
  (a) $\{(t^2, t^3) \mid t \in \mathbb{C}\}$
  (b) $\{(t^2, t^3 + 1) \mid t \in \mathbb{C}\}$
  (c) $\{(t^2, t^2) \mid t \in \mathbb{C}\}$
  (d) $\{(t, t) \mid t \in \mathbb{C}\} \cup \{(t, -t) \mid t \in \mathbb{C}\}$
  (e) $\{(t^3 + 1, t^2 + t) \mid t \in \mathbb{C}\}$

(2) Describe the following varieties:
  (a) $\mathbb{V}(X^2 - Y, X^3 - Z) \subseteq \mathbb{C}^3$
  (b) $\mathbb{V}(X^2 - YZ, XZ - X) \subseteq \mathbb{C}^3$
  (c) $\mathbb{V}(XZ - Y^2, Z^3 - X^5) \subseteq \mathbb{C}^3$

(3) We say that $\mathbf{x} = (x, y) \in \mathbb{C}^2$ is an *interior point* of a subset $A \subseteq \mathbb{C}^2$ if there exists some ball $B(\mathbf{x}, r)$ centred on $\mathbf{x}$ of radius $r > 0$ such that

$$B(\mathbf{x}, r) \subset A.$$

Show that a non-trivial variety

$$V \subset \mathbb{C}^2$$

cannot have interior points.

(4) Show that the set

$$V = \{(x, y) \in \mathbb{C}^2 \mid y = \mathrm{e}^x\}$$

is *not* a variety.

Solutions

(1) (a) We have that

$$(t^2)^3 = t^6 = (t^3)^2$$

So writing $X = t^2$ and $Y = t^3$, we can write this set as

$$\mathbb{V}(Y^2 - X^3) \subseteq \mathbb{C}^2.$$

(b) Similarly to part (a), if we write $X = t^2$ and $Y = t^3 + 1$, then $X^3 = (Y - 1)^2 = t^6$, and hence we can write the set as

$$\mathbb{V}((Y - 1)^2 - X^3) \subseteq \mathbb{C}^2.$$

(c) Note that for any $s \in \mathbb{C}$, we can find $t \in \mathbb{C}$ such that $t^2 = s$, and hence this set can be rewritten as

$$\{(s, s) \mid s \in \mathbb{C}\} = \mathbb{V}(Y - X) \subseteq \mathbb{C}^2.$$

(d) We have

$$\{(t,t) \mid t \in \mathbb{C}\} = \mathbb{V}(Y - X),$$

$$\{(t,-t) \mid t \in \mathbb{C}\} = \mathbb{V}(Y + X).$$

Then using the fact that $\mathbb{V}(F) \cup \mathbb{V}(G) = \mathbb{V}(F \cdot G)$, we have that

$$\{(t,t) \mid t \in \mathbb{C}\} \cup \{(t,-t) \mid t \in \mathbb{C}\} = \mathbb{V}(Y^2 - X^2).$$

(e) This one is a little more difficult to describe as a zero set. One method of doing so would be to write $X = t^3 + 1$ and $Y = t^2 + t$, and then take various products and multiples:

$$\begin{array}{ccccccccccccccc}
1 & = & 1 & & & & & & & & & & & & \\
Y & = & & & t & + & t^2 & & & & & & & & \\
Y^2 & = & & & & & t^2 & + & 2t^3 & + & t^4 & & & & \\
Y^3 & = & & & & & & & t^3 & + & 3t^4 & + & 3t^5 & + & t^6 \\
X & = & 1 & & & & & + & t^3 & & & & & & \\
X^2 & = & 1 & & & & & + & 2t^3 & & & & & + & t^6 \\
XY & = & & & t & + & t^2 & & & + & t^4 & + & t^5 & &
\end{array}$$

We can rewrite this as a matrix equation:

$$\begin{bmatrix} 1 \\ Y \\ Y^2 \\ Y^3 \\ X \\ X^2 \\ XY \end{bmatrix} = \begin{bmatrix} 1 & & & & & & \\ & 1 & 1 & & & & \\ & & 1 & 2 & 1 & & \\ & & & 1 & 3 & 3 & 1 \\ 1 & & & 1 & & & \\ 1 & & & 2 & & & 1 \\ & 1 & 1 & & 1 & 1 & \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \end{bmatrix}$$

Then we can do row reduction to get a row of zeros and hence a solution: for this, we have

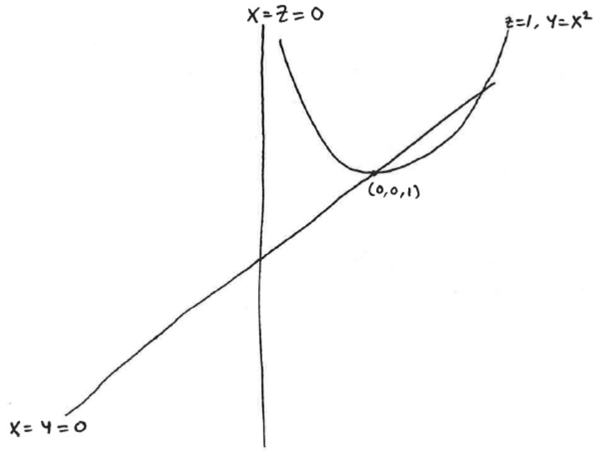
$$Y^3 - X^2 - 3XY + X + 3Y = 0,$$

and hence the set is $\mathbb{V}(Y^3 - X^2 - 3XY + X + 3Y) \subseteq \mathbb{C}^2$.

(2) (a) Note that from the first polynomial we have $Y = X^2$ and from the second we have $Z = X^3$. So if we use a parameter $t \in \mathbb{C}$, we can write

$$\mathbb{V}(X^2 - Y, X^3 - Z) = \{(t, t^2, t^3) \mid t \in \mathbb{C}\}.$$

(b) From the second polynomial, we have $X(Z - 1) = 0$. So we must have $X = 0$ or $Z = 1$ (or both).
If $X = 0$, then the first polynomial gives $YZ = 0$, which is satisfied when $Y = 0$ or $Z = 0$: so the lines $X = Y = 0$ and $X = Z = 0$ are part of the variety.
If $Z = 1$, then the first polynomial gives $Y = X^2$, and hence the parabola $Z = 1, Y = X^2$ is another part of the variety.



(c) First use both polynomials to note that the only points in this variety with $X = 0$ or $Y = 0$ are the origin $(0, 0, 0)$. So assume that $X \neq 0, Y \neq 0$.
We can then look at the equality from the first polynomial:

$$XZ - Y^2 = 0 \Leftrightarrow \frac{Y}{X} = \frac{Z}{Y} =: t.$$

(We can divide by X and Y since we assumed they're non-zero.) So we have

$$Y = tX,\ Z = tY = t^2X,$$

and hence

$$\mathbb{V}(XZ - Y^2) = \{(x, tx, t^2x) \mid x, t \in \mathbb{C}\}.$$

To calculate the first variety, we substitute these points into the second polynomial (as we want the intersection:

$$t^6x^3 - x^5 = 0.$$

We can cancel a factor of x^2 (again, since $x \neq 0$), and get

$$x^2 = t^6.$$

Then this is only true if $x = t^3$ or $x = -t^3$. So our variety V has two pieces:

$$V = \{(t^3, t^4, t^5) \mid t \in \mathbb{C}\} \cup \{(-t^3, -t^4, -t^5) \mid t \in \mathbb{C}\}.$$

(3) First note that it suffices to show this for a variety of the form $\mathbb{V}(F)$, since the intersection of two sets without interior points can't gain interior points.
Suppose $P = (x, y)$ is an interior point of

$$V = \mathbb{V}(F(X, Y)) \subseteq \mathbb{C}^2.$$

Fix $X = x \in \mathbb{C}$. Then we have a polynomial $F_x(Y) = F(x, Y) \in \mathbb{C}[Y]$. Since (x, y) is an interior point, we have that $F_x(Y) = 0$ on the set $(y - r, y + r)$: but we know that a non-zero polynomial in one variable can only have finitely many zeros. So $F_x(Y)$ is identically zero.

Similarly, the polynomial $F_y(X) \in \mathbb{C}[X]$ is identically zero, and hence $F(X,Y)$ is identically zero. So $\mathbb{V}(F) = \mathbb{C}^2$, giving us a contradiction (we assumed V was non-trivial).

(4) Suppose that V were a variety. Then the set

$$W = V \cap \mathbb{V}(Y-1)$$

would also be a variety (since we know that the intersection of two varieties is a variety).

We have

$$W = \{(x,1) \mid \mathrm{e}^x = 1\} = \{(2\pi k i, 1) \mid k \in \mathbb{Z}\}.$$

Then we just need to show the following: a variety $V \subseteq \mathbb{C}^2$ can only have finitely many isolated points (i.e. points $\mathbf{x}$ of V with the property that there exists a ball $B(\mathbf{x}, r)$ around $\mathbf{x}$ such that $B(\mathbf{x}, r) \cap V = \{\mathbf{x}\}$). Note that all points of W are isolated, and that it does have infinitely many points: so if we can prove this result, then neither W nor V can be varieties.

First, we note that $\mathbb{V}(F) \subseteq \mathbb{C}^2$ cannot have *any* isolated points. Then we reason that by Bézout's theorem (mentioned in Lecture 1), the intersection $\mathbb{V}(F, G)$ of two zero sets must have finitely many isolated points. Adding in further polynomials only brings down the number of intersection points: so infinitely many will be impossible.